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ESTIMATES OF THE PROXIMITY OF GAUSSIAN MEASURES

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Let H be a separable Hilbert space with inner product $(\cdot, \cdot)$ and norm $|\cdot|$. Let N_A^a be the Gaussian measure on H with mean value a and correlation operator A.

According to the Feldman-Hájek theorem (see, for example, [1], Chapter II, §3), two Gaussian measures on H are either equivalent or orthogonal.

Theorems 1 and 2 below yield two-sided estimates for the variation of the difference of two equivalent Gaussian measures. Theorems 3–5 give uniform estimates of the proximity of Gaussian measures over the class of all balls in H with a fixed center (Theorems 3 and 4) and with an arbitrary center (Theorem 5) for the general case when the Gaussian measures can be orthogonal.

Let $\|M\|$ denote the variation of a generalized measure M. Since $\|N_A^a - N_B^b\| = \|N_A^{a-b} - N_B^0\|$, in determining estimates of the proximity in variation of equivalent measures it can be assumed that the mean value of one of the measures is zero. Everywhere below, $N_B = N_B^0$.

It is known (see [1], Chapter II, §3, Theorem 3.1) that the measures N_A^a and N_A are equivalent if and only if $a \in A^{1/2}(H)$, where $A^{1/2}$ is the positive square root of the operator A.

THEOREM 1. *If the measures N_A^a and N_A are equivalent, then $\|N_A^a - N_A\| = 2\varphi(|h|)$, where $h = A^{-1/2}a$ and $\varphi(u) = (2\pi)^{-1/2}\int_{-u/2}^{u/2}\exp(-t^2/2)\,dt$ for $u \geq 0$.*

To prove Theorem 1 we use the known equality (see, for example, [1], Chapter II, §3, Theorem 3.1)

$$(dN_A^a/dN_A)(x) = \exp\{(A^{-1/2}x, h) - |h|^2/2\},$$

which implies that

$$\|N_A^a - N_A\| = 2E(\exp(Y - |h|^2/2) - 1) \cdot 1_{\{Y \geq |h|^2/2\}},$$

where $1_{\{D\}}$ is the indicator function of the set D, and the random variable (RV) Y has normal distribution with mean 0 and variance $|h|^2$.

COROLLARY 1. *If the measures N_A^a and N_A are equivalent, then*

$$8\sqrt{2}|h|/(\sqrt{\pi}(8 + |h|^2)) \leq \|N_A^a - N_A\| \leq \sqrt{2}|h|/\sqrt{\pi}.$$

It is known (see, for example, [2], Chapter VII, §4, Theorem 2) that N_A and N_B are equivalent if and only if $B = A^{1/2}TA^{1/2}$, where T is a positive-definite operator such that $T - I$ is a Hilbert-Schmidt operator; here I is the identity operator.

THEOREM 2. *If the measures N_A and N_B are equivalent, then*

$$\|N_A - N_B\| = \tfrac{1}{2}E|Y| + R,$$

where $Y = \sum_1^\infty d_i Y_i$, $(d_i)_1^\infty$ are the eigenvalues of the operator $T - I$, $Y_i = X_i^2 - 1$, $i = 1, 2, \ldots$, and $(X_i)_1^\infty$ are independent normally distributed RV's with mean 0 *and*

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variance 1. *Further, if* $d = (\sum_1^\infty d_i^2)^{1/2} < 1$, *then the remainder* R *admits the estimates*

$$-d^2 \ln d^{-1} - 4d^2 - \sqrt{2} q d/4 \le R \le q, \tag{1}$$

where $q = \sum_1^\infty (d_i - \ln(1 + d_i))$.

In the proof of Theorem 2, using the equality (see [2], Chapter VII, §4, equality (2))

$$\frac{dN_A}{dN_B}(x) = \exp\left\{-\frac{1}{2}\sum_1^\infty ((A^{-1/2}x, e_i)^2 d_i (1+d_i)^{-1} - \ln(1+d_i))\right\},$$

where the e_i are eigenvectors of the operator $T - I$ corresponding to the eigenvalues d_i, we have that

$$\|N_A - N_B\| = 2E(1 - \exp(-(Y+q)/2)) \cdot 1_{\{Y+q>0\}}.$$

This easily leads to the upper estimate for R in (1). The lower estimate for R in (1) follows from the relations

$$\begin{aligned}\|N_A - N_B\| &\ge 2E(1 - \exp(-(Y+q)/2)) \cdot 1_{\{q<Y+q<q-2\sqrt{2}d\ln d\}} \\ &\ge E(Y+q)\exp(-(q - 2\sqrt{2}d\ln d)/2) \cdot 1_{\{0<Y<-2\sqrt{2}d\ln d\}}\end{aligned}$$

and the Tchebycheff inequality.

Using the Hölder inequality, we get that $\sqrt{2/15}d \le E|Y| \le \sqrt{2}d$. Therefore, Theorem 2 gives us

COROLLARY 2. *If* $d \le 0.02$, *then* $0.02d \le \|N_A - N_B\| \le d$.

Theorems 1 and 2 combined lead to estimates for $\|N_A^a - N_B^b\|$ in the general case. In particular, it is possible to improve the estimates obtained in [3] for the proximity of Gaussian measures on the class of convex Borel sets in the k-dimensional Euclidean space $\mathbf{R}^k$.

COROLLARY 3. *Suppose that the measures* N_A^a *and* N_B^b *on* $\mathbf{R}^k$ *have nonsingular covariance matrices* A *and* B, *respectively. If there exists an* ε, $0 < \varepsilon < 1$, *such that* $|(Bx, x) - (Ax, x)| \le \varepsilon(Ax, x)$ *for all* $x \in \mathbf{R}^k$, *then*

$$\|N_A^a - N_B^b\| \le \sqrt{2/\pi}|A^{-1/2}(a-b)| + 1.82k^{1/2}\varepsilon(1-\varepsilon)^{-1/2}. \tag{2}$$

REMARK. It follows from Corollary 2 that the exponent of k in (2) cannot be decreased.

We proceed to an estimate of the proximity of the measures N_A^a and N_B^b on the balls

$$S_r(d) = \{x \in H : |x - d| < r, d \in H, r > 0\}.$$

Since

$$N_A^a(S_r(d)) = N_A^{a-d}(S_r(0)), \tag{3}$$

it suffices in the case when $a \ne b$ and $A = B$ to confine oneself to balls about zero. Below, $S_r = S_r(0)$.

THEOREM 3. *There exists an absolute constant* c *such that*

$$|(N_A^a - N_B^b)(S_r)| \le c c(A)((\operatorname{tr} A)^{1/2} + |a| + |b|)|a - b|,$$

where $\operatorname{tr} A$ *is the trace of the operator* A,

$$c(A) = \min\left\{a_1^{-1}a_2^{-1}\left(\prod_1^3 f_i(A)\right)^{-2/3}\right\},$$

$f_j^4(A) = \sum_{i \geq j} a_i^4$ for $j = 1, 2, \ldots$, and $a_1^2 \geq a_2^2 \geq \cdots$ are the eigenvalues of A.

To prove Theorem 3 we should observe that

$$|(N_A^a - N_A^b)(S_r)| \leq (N_A^a + N_A^b)(S_{r+|a-b|} \setminus S_r),$$

and then construct estimates for the density of the RV $|Y|$ for a random element Y having the Gaussian distribution N_A^a. To do this we should use Lemma 6 in [4], the inversion formula, and the equality (see [5], p. 82)

$$E \exp\{it|Y|^2\} = \exp\{it(R_A a, a)\} \prod_1^\infty (1 - 2ita_j^2)^{-1/2}, \tag{4}$$

where $R_A = (I - 2itA)^{-1}$.

In the case when the mean values of the measures copincide, it can be assumed in view of (3) that the mean values of the measures are zero, but balls with nonzero centers will be considered. We introduce the following notation:

$$D = A - B, \quad |A| = \sup_{|x|=1} |Ax|, \quad c_i(A) = \left(\prod_1^i f_j(A) \right)^{(1-i)/i},$$

$$c_i = (|A| + |B|)^{(i-1)/2} (c_i(A) + c_i(B))$$

for $i = 3, 5, 7, 9$, where the quantities $f_j(A)$ are the same as in Theorem 3; and

$$T = ((|A| + |B|)^2 + (Aa, a) + (Ba, a))^{-1/2}.$$

THEOREM 4. a) *There exists a constant c such that*

$$\Delta(a) = \sup_{r>0} |(N_A - N_B)(S_r(a))|$$
$$\leq c(c_9(T|D|)^{1/2} + c_3 T \operatorname{tr}(D^2)^{1/2} + c_5 T^2 |(Da, a)|).$$

b) *If $AB = BA$, then*

$$\Delta(a) \leq c(c_7 T|D| + c_3 T \operatorname{tr}(D^2)^{1/2} + c_5 T^2 |(Da, a)|).$$

Using the differentiability of the function $g(s,t) = E \exp\{it|X - a|^2\}$, with respect to $s \in [0,1]$, where the random element X has distribution N_V with $V = A + s(B - A)$, the Esseen inequality, and the representation (4), we get in the proof of Theorem 4 that

$$2\pi\Delta(a) \leq \int_0^1 \int_{-\infty}^{\infty} |g(s,t)| \cdot |2it(R_V D R_V a, a) + \operatorname{tr}(DR_V)| \, dt \, ds.$$

Further,

$$|g(s,t)| \leq f(t) \exp\{-2t^2 (Va, a)/(1 + 4t^2 |V|^2)\},$$

where $f(t) = \prod_1^\infty (1 + 4t^2 v_j^4)^{-1/4}$ and $v_1^2 \geq v_2^2 \geq \cdots$ are the eigenvalues of the operator V. Since $R_V = I + 2itVR_V$,

$$|(R_V D R_V a, a)| \leq |(Da, a)| + 4|t| \, |DV|^{1/2} ((A + B)a, a) + 4t^2 |DV| (Va, a).$$

If $AB = BA$, then

$$|(R_V D R_V a, a)| = |(R_V^2 Da, a)| \leq |(Da, a)| + 4|t| \cdot |D| (Va, a).$$

REMARK. For any unit vector $e \in H$ with $(Ae, e) + (Be, e) > 0$ there exists a constant c such that (see [6])

$$\sup_{r>0} |(N_A - N_B)x \in H \colon \{(x, e) < r\}| \leq c|(De, e)|/((A + B)e, e).$$

An estimate of the same kind can be derived from Theorem 4 by using the relation (see [5], p. 70)

$$\{x\colon (x,e) < r\} = \bigcup_{1}^{\infty} \{x\colon |x + (n-r)e| < n\}. \tag{5}$$

If additional information about the connection between A and B is known, then it is possible to get sharper estimates than in Theorem 4.

THEOREM 5. a) *If the kernels of the operators A and B coincide, i.e.,* $\ker A = \ker B$, *then*

$$\Delta = \sup_{a\in H} \Delta(a) \geq \tfrac{5}{8}(3c_5 + \operatorname{tr}(A+B)(c_3(A) + c_3(B)))|A^{-1/2}BA^{-1/2} - I|.$$

b) *If* $\ker A \neq \ker B$, *then* $\Delta \geq 1/2$.

The proof of Theorem 5a) is analogous to that of Theorem 4. Let V and R_V be the same as above, and let $d = |A^{-1/2}BA^{-1/2} - I|$. Without loss of generality let $d < 1$. It is easy to show that $|G| \leq d/(1-d)$ for $G = V^{-1/2}DV^{-1/2}$. Therefore,

$$|(R_V D R_V a, a)| = |(R_V G R_V V^{1/2}a, V^{1/2}a)| \leq d(Va,a)/(1-d)$$

and

$$|\operatorname{tr}(DR_V)| \leq |G| \operatorname{tr} V \leq d \operatorname{tr} V/(1-d).$$

Theorem 5b) follows from (5), with $r = 0$ and a vector e belonging to only one of the spaces $\ker A$ and $\ker B$.

COROLLARY 4. *If $B = \lambda A$ and $\lambda > 1$, then*

$$\Delta \leq (15.9|A|^2 c_5(A) + 2.6\operatorname{tr}(A))(\lambda - 1).$$

REMARK. 1. It is not hard to show that if $B = \lambda A$ and $0 < \lambda - 1 < \varepsilon$, then for all sufficiently small ε there exists a constant c such that $\Delta \geq c(\lambda - 1)$.

2. Corollary 4 was used in [7] to study estimates of the rate of convergence in the central limit theorem for sums of a random number of random elements in H.

3. A number of problems use characteristics of proximity of Gaussian measures different from those considered in this article (see, for example, [8]). The results in [8] were used in [9] to construct estimates of the proximity of Gaussian measures on balls in H. The estimates in [9] do not indicate the explicit dependence of the constants determined by the correlation operators of the measures on the spectra of these operators. The estimates in [9] are also inferior to those of the present article with respect to the dependence on the center a of the ball.

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