Stability boundary approximation of periodic dynamics

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Motivation

Stability of linear dynamical system with periodic parameters

$$\dot{x}(t) = \mathbf{J}(t)\, x(t),$$

$x(t)$ is the state variable vector
$\mathbf{J}(t)$ is a T-periodic $\mathbf{J}(t) = \mathbf{J}(t+T)$, piecewise continuous, and integrable matrix.

Stability of periodic solutions of nonlinear dynamical system

Lyapunov second method amounts to checking the stability of linearized system

$$\dot{x}(t) = \mathbf{J}(t)\, x(t),$$

$x(t)$ is the vector of perturbations corresponding to the state variables
$\mathbf{J}(t)$ is the *Jacobian matrix* of an originally nonlinear system.

Stability of linear systems with periodic coefficients

G. Floquet,
Sur les équations différentielles linéaires à coefficients périodiques.
Annales scientifiques de l'École normale supérieure 12 (1883) 47–88.

General stability, Floquet (1883)

$$\dot{\mathbf{X}}(t) = \mathbf{J}(t) \cdot \mathbf{X}(t), \quad \mathbf{X}(0) = \mathbf{I}. \tag{1}$$

Monodromy matrix **F** is the fundamental matrix at time $t = T$:

$$\mathbf{F} = \mathbf{X}(T)$$

Floquet multipliers $\rho = \mathrm{eig}(\mathbf{F})$ are the eigenvalues of the monodromy matrix.

$$|\rho_i| < 1 \quad \forall\, i \iff \text{asymptotic stability.}$$
$$\exists\, i : |\rho_i| > 1 \implies \text{instability.}$$

Excitation smallness assumption

Let there exists autonomous matrix $\mathbf{J}_0$, such that the residual $\mathbf{J}(t) - \mathbf{J}_0$ is small and can be represented as the series

$$\mathbf{J}(t) - \mathbf{J}_0 = \mathbf{J}_1(t) + \mathbf{J}_2(t) + \mathbf{J}_3(t) + \dots,$$

where the lower index denotes the order of smallness.

Transformation

The change of variables $\mathbf{X}(t) = \exp(\mathbf{J}_0 t) \cdot \mathbf{Y}(t)$ converts matrix differential equation (1) into the form:

$$\dot{\mathbf{Y}}(t) = \mathbf{H}(t) \cdot \mathbf{Y}(t), \quad \mathbf{Y}(0) = \mathbf{I}, \tag{2}$$

where matrix

$$\mathbf{H}(t) := \exp(-\mathbf{J}_0 t) \cdot (\mathbf{J}(t) - \mathbf{J}_0) \cdot \exp(\mathbf{J}_0 t)$$

is small.

Approximate solution of (2) can be found with the use of *averaging scheme*.

Averaging

Let

$$\mathbf{H}(t) = \mathbf{H}_1(t) + \mathbf{H}_2(t) + \mathbf{H}_3(t) + \dots, \tag{3}$$

where $\mathbf{H}_j(t) := \exp(-\mathbf{J}_0 t) \cdot \mathbf{J}_j(t) \cdot \exp(\mathbf{J}_0 t)$.
We will find solution in the form

$$\mathbf{Y}(t) = (\mathbf{I} + \mathbf{U}_1(t) + \mathbf{U}_2(t) + \dots) \cdot \mathbf{Z}(t), \tag{4}$$

where $\mathbf{U}_j(t)$ are T-periodic matrix-functions, such that $\mathbf{U}_j(0) = \mathbf{U}_j(T) = 0$, and $\mathbf{Z}(t)$ is the solution of the *averaged differential equation*:

$$\dot{\mathbf{Z}}(t) = (\mathbf{A}_1 + \mathbf{A}_2 + \mathbf{A}_3 + \dots) \cdot \mathbf{Z}(t), \quad \mathbf{Z}(0) = \mathbf{I}, \tag{5}$$

which can be written via the matrix exponential $\mathbf{Z}(t) = \exp\left([\mathbf{A}_1 + \mathbf{A}_2 + \mathbf{A}_3 + \dots]\, t\right)$. Hence due to $\mathbf{Y}(T) = \mathbf{Z}(T)$

$$\mathbf{F} = \mathbf{X}(T) = \mathbf{F}_0 \cdot \mathbf{Y}(T) = \mathbf{F}_0 \cdot \mathbf{Z}(T), \tag{6}$$

where we denote $\mathbf{F}_0 := \exp(\mathbf{J}_0 T)$ as zero order approximation of monodromy matrix, $\mathbf{F} \approx \mathbf{F}_0$.

Averaging

$$\dot{\mathbf{Y}}(t) = \mathbf{H}(t) \cdot \mathbf{Y}(t), \quad \mathbf{Y}(0) = \mathbf{I}, \tag{2}$$

$$\mathbf{H}(t) = \mathbf{H}_1(t) + \mathbf{H}_2(t) + \mathbf{H}_3(t) + \dots, \tag{3}$$

$$\mathbf{Y}(t) = (\mathbf{I} + \mathbf{U}_1(t) + \mathbf{U}_2(t) + \dots) \cdot \mathbf{Z}(t), \tag{4}$$

$$\dot{\mathbf{Z}}(t) = (\mathbf{A}_1 + \mathbf{A}_2 + \mathbf{A}_3 + \dots) \cdot \mathbf{Z}(t), \quad \mathbf{Z}(0) = \mathbf{I}, \tag{5}$$

The matrices $\mathbf{A}_j$ and matrix-functions $\mathbf{U}_j(t)$ can be found one by one substituting expressions for time derivatives from (3)–(5) into (2), collecting there terms of the same order and canceling non-degenerate matrix $\mathbf{Z}$.

Monodromy matrix

$$\mathbf{F} = \mathbf{F}_0 \cdot \exp\left([\mathbf{A}_1 + \mathbf{A}_2 + \mathbf{A}_3 + \dots]\, T\right), \tag{6}$$

где $\mathbf{F}_0 := \exp(\mathbf{J}_0 T)$.

Zero approximation of monodromy matrix

$$\mathbf{F}_0 := \exp(\mathbf{J}_0 T).$$

First order approximation of monodromy matrix

$$\mathbf{F} \approx \mathbf{F}_0 \cdot (\mathbf{I} + \mathbf{A}_1 T)$$

where

$$\mathbf{A}_1 = \frac{1}{T} \int_0^T \mathbf{H}_1(t)\,\mathrm{d}t \tag{7}$$

and $\mathbf{H}_1(t) := \exp(-\mathbf{J}_0 t) \cdot \mathbf{J}_1(t) \cdot \exp(\mathbf{J}_0 t)$.
Hence

$$\mathbf{F} \approx \exp(\mathbf{J}_0 T) \cdot \left(\mathbf{I} + \int_0^T \exp(-\mathbf{J}_0 t) \cdot \mathbf{J}_1(t) \cdot \exp(\mathbf{J}_0 t)\,\mathrm{d}t \right).$$

Second order approximation of monodromy matrix

$$\mathbf{U}_1(t) = \int_0^t (\mathbf{H}_1(\tau) - \mathbf{A}_1)\, \mathrm{d}\tau,$$

$$\mathbf{A}_2 = \frac{1}{T}\int\limits_0^T (\mathbf{H}_2(t) + \mathbf{H}_1(t)\cdot\mathbf{U}_1(t) - \mathbf{U}_1(t)\cdot\mathbf{A}_1)\,\mathrm{d}t, \tag{8}$$

where we use matrix $\mathbf{A}_1$, calculated (7),
$\mathbf{H}_2(t) := \exp(-\mathbf{J}_0 t)\cdot\mathbf{J}_2(t)\cdot\exp(\mathbf{J}_0 t)$.
Expansion of the matrix exponential in (6) up to the second order terms yields

$$\mathbf{F} \approx \mathbf{F}_0 \cdot \left(\mathbf{I} + \mathbf{A}_1 T + \mathbf{A}_2 T + \frac{1}{2}\mathbf{A}_1^2 T^2\right).$$

Third order approximation of monodromy matrix

Using $\mathbf{A}_2$ and $\mathbf{U}_1(t)$ from (8), we calculate:

$$\mathbf{U}_2(t) = \int_0^t (\mathbf{H}_2(\tau) - \mathbf{A}_2 + \mathbf{H}_1(\tau)\cdot\mathbf{U}_1(\tau) - \mathbf{U}_1(\tau)\cdot\mathbf{A}_1)\,\mathrm{d}\tau,$$

$$\mathbf{A}_3 = \frac{1}{T}\int_0^T (\mathbf{H}_3(t) + \mathbf{H}_2(t)\cdot\mathbf{U}_1(t) - \mathbf{U}_1(t)\cdot\mathbf{A}_2 + \mathbf{H}_1(t)\cdot\mathbf{U}_2(t) - \mathbf{U}_2(t)\cdot\mathbf{A}_1)\,\mathrm{d}t.$$

where $\mathbf{H}_3(t) := \exp(-\mathbf{J}_0 t)\cdot\mathbf{J}_3(t)\cdot\exp(\mathbf{J}_0 t)$.

$$\mathbf{F} \approx \mathbf{F}_0 \cdot \Big(\mathbf{I} + \mathbf{A}_1 T + \mathbf{A}_2 T + \frac{1}{2}\mathbf{A}_1^2 T^2 + \mathbf{A}_3 T + \frac{1}{2}(\mathbf{A}_1\cdot\mathbf{A}_2 + \mathbf{A}_2\cdot\mathbf{A}_1)\,T^2 + \frac{1}{6}\mathbf{A}_1^3 T^3\Big),$$

and so on ...

Derivatives of monodromy matrix with respect to parameters

Similar relations one can obtain with the use of derivatives (Seyranian, Solem, Pedersen (1999) Arch.Appl.Mech.)

$$\frac{\partial \mathbf{F}}{\partial p_j} = \mathbf{F} \int_0^T \mathbf{X}^{-1} \frac{\partial \mathbf{J}}{\partial p_j} \mathbf{X} \, \mathrm{d}t.$$

Inverted pendulum with oscillating pivot

Seyranian, A.A., Seyranian, A.P.
The stability of an inverted pendulum with a vibrating suspension point.
Journal of Applied Mathematics and Mechanics **70**, 754–761 (2006)

Pendulum with vertically oscillating pivot I

Nonlinear system

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\left(m\,l^2\frac{\mathrm{d}\theta}{\mathrm{d}\tau}\right)+\gamma l^2\frac{\mathrm{d}\theta}{\mathrm{d}\tau}+m\,l\left(g+a\frac{\mathrm{d}^2}{\mathrm{d}\tau^2}\varphi(\Omega\tau)\right)\sin\theta=0$$

l – length
m – mass,
φ – 2π-periodic twice-differentiable function,
a – amplitude of pivot oscillations,
$\Omega_0=\sqrt{\frac{g}{l}}$ – frequency, a $T_0=2\pi/\Omega_0$ – period of small oscillations without excitation,
γ – viscous friction coefficient.

Inverted pendulum with oscillating pivot I

Three dimentionless parameters

$$\varepsilon = \frac{a}{l}, \quad \omega = \frac{\Omega_0}{\Omega} = \frac{T}{T_0}, \quad \beta = \frac{\gamma}{m\Omega_0},$$

Dimentionless equations (in new time $t = \Omega\tau$)

$$\dot{\theta} = s, \quad \dot{s} = -\beta\omega s - \left(\omega^2 + \varepsilon\ddot{\varphi}(t)\right)\sin(\theta),$$

where s – angular velocity, dot is the derivative w.r.t. t.

Equations linearized about $(\theta, s) = (\pi, 0)$

$$\dot{x}(t) = \mathbf{J}(t)\,x(t), \quad \mathbf{J}(t) = \begin{pmatrix} 0 & 1 \\ \omega^2 + \varepsilon\ddot{\varphi}(t) & -\beta\omega \end{pmatrix}.$$

Inverted pendulum with oscillating pivot II

Multipliers

Eigenvalues ρ_1 и ρ_2 of monodromy matrix are defined by characteristic polynomial:

$$\rho^2 - \mathrm{tr}(\mathbf{F})\,\rho + \det(\mathbf{F}) = 0. \tag{1}$$

Stability conditions ($|\rho_1| \leq 1$ и $|\rho_2| \leq 1$)

For real roots $\rho \in [-1, 1]$, and for complex conjugate roots $\rho_1\rho_2 \leq 1$. From (1) and Vieta's formula $\rho_1\rho_2 = \det(\mathbf{F})$ we obtain:

$$|\,\mathrm{tr}(\mathbf{F})| \leq 1 + \det(\mathbf{F}) \quad \text{и} \quad \det(\mathbf{F}) \leq 1, \tag{2}$$

where for asymptotic stability all inequalities should be strict.

Inverted pendulum with oscillating pivot III

Express matrix as series $\mathbf{J}(t) = \mathbf{J}_0 + \mathbf{J}_1(t) + \mathbf{J}_2$, where

$$\mathbf{J}_0 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \mathbf{J}_1(t) = \begin{pmatrix} 0 & 0 \\ \varepsilon\ddot{\varphi}(t) & 0 \end{pmatrix}, \quad \mathbf{J}_2 = \begin{pmatrix} 0 & 0 \\ \omega^2 & -\beta\omega \end{pmatrix},$$

assuming ε, ω, and β small of the same order.

$$\exp(\mathbf{J}_0 t) = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}, \quad \exp(-\mathbf{J}_0 t) = \begin{pmatrix} 1 & -t \\ 0 & 1 \end{pmatrix}.$$

Zero approximation

$$\mathbf{F} \approx \exp(\mathbf{J}_0 2\pi) = \begin{pmatrix} 1 & 2\pi \\ 0 & 1 \end{pmatrix}.$$

Inverted pendulum with oscillating pivot IV

First approximation

$$\mathbf{F} \approx \begin{pmatrix} 1 & 2\pi \\ 0 & 1 \end{pmatrix} + \varepsilon \begin{pmatrix} \pi^2 & 0 \\ 0 & -\pi^2 \end{pmatrix}.$$

Second approximation

$$\begin{aligned} \mathbf{F} \approx & \begin{pmatrix} 1 & 2\pi \\ 0 & 1 \end{pmatrix} + \varepsilon \begin{pmatrix} \pi^2 & 0 \\ 0 & -\pi^2 \end{pmatrix} \\ & + \begin{pmatrix} -\frac{1}{6}\pi^4\varepsilon^2 + 2\pi^2\omega^2 & -\frac{1}{15}\pi^5\varepsilon^2 + \frac{4}{3}\pi^3\omega^2 - 2\pi^2\beta\omega \\ -\frac{2}{3}\pi^3\varepsilon^2 + 2\pi\omega^2 & -\frac{1}{6}\pi^4\varepsilon^2 + 2\pi^2\omega^2 - 2\pi\beta\omega \end{pmatrix}. \end{aligned}$$

Stability boundaries in the second and third approximations

$$\varepsilon_p = \frac{2\sqrt{3}}{\pi}\,\omega,$$

$$\varepsilon_n = \frac{2\sqrt{3}}{\pi}\sqrt{\omega^2 - \frac{\beta\omega}{\pi} + \frac{1}{\pi^2}}.$$

Stability boundaries in the fourth approximation

$$\frac{\pi^8\varepsilon^4}{1260}-\frac{\pi^4}{3}\left(1+\frac{4\pi^2\omega^2}{15}-\pi\beta\omega\right)\varepsilon^2+4\pi^2\omega^2\left(1+\frac{\pi^2\omega^2}{3}-\beta\omega\pi\right)=0,$$

$$\frac{\pi^8\varepsilon^4}{1260}-\frac{\pi^4}{3}\left(1+\frac{4\pi^2\omega^2}{15}-\pi\beta\omega\right)\varepsilon^2+4\pi^2\omega^2\left(1+\frac{\pi^2\omega^2}{3}-\beta\omega\pi\right)+$$
$$+4\left(1-\beta\pi\omega+\pi^2\omega^2\beta^2\right)=0$$

Comparison of approximate and exact stability boundaries

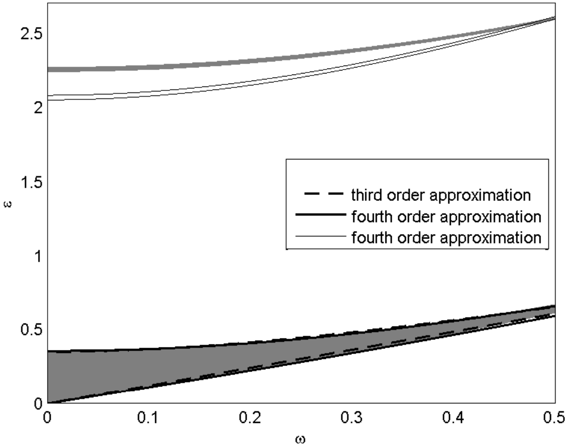

Figure: Stability boundaries in the third approximation (dashed lines) and the fourth approximation (solid lines) in comparison with exact stability domains (gray).

Damping stabilization and destabilization of inverted vertical pendulum position

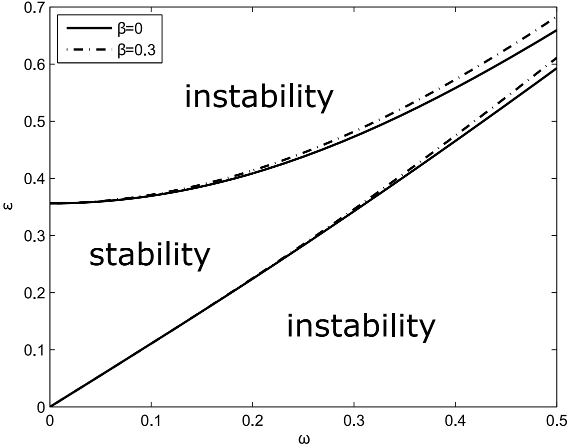

Figure: Addition of small linear viscous friction β shifts both stability boundaries upward. Thus, at the lower boundary additional friction destabilizes the inverted pendulum while at the upper boundary friction stabilizes the pendulum position.

Discussion on accuracy of stability borders

For instance, the eigenvalue problem with the matrix:

$$\mathbf{J} = \begin{pmatrix} 0 & 1 \\ O(\varepsilon) & 0 \end{pmatrix}$$

has the characteristic equation

$$\lambda^2 - O(\varepsilon) = 0,$$

so that the double zero eigenvalue,

$$\lambda = O(\sqrt{\varepsilon}),$$

is determined up to a small summand of the order of smallness half of that of the matrix.
Hence, n-multiplicity of eigenvalue can decrease the order of its approximation at most n-times with respect to the order of approximation of the matrix.