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NONLINEAR OPTICS

Non-One-Dimensional Optical Solitons and Ideal Fluid Dynamics

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Abstract—Influence of transverse effects on the propagation of light beams in gradient glass fiber and on the pulsed mode of the second harmonic generation is investigated. Within the approaches under consideration the equations for the pulse or beam field envelopes are reduced to a system of hydrodynamic-type equations for amplitudes and eikonals. These equations are used to describe the vortex and non-vortex beam channeling modes and the propagation of light bullets.

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1. INTRODUCTION

The notion of the soliton has penetrated into almost all fields of physics, being probably most widely used in optics. There are spatial and temporal optical solitons. Spatial solitons are optical beams almost unlimited in the direction of their propagation and limited in transverse directions. These solitons occur when the self-focusing of the beam is compensated for by its diffraction due to nonlinearity of the medium [1]. Temporal solitons are optical pulses of finite duration unlimited in transverse directions and limited in the direction of propagation. Here the nonlinear breaking of the pulse is compensated for by its dispersion.

In its strict sense, the term soliton usually refers to a solitary wave capable of elastically interacting with others like it. By solitons are often meant (especially in nonlinear optics) simply solitary waves that are not necessarily required to interact elastically with one another. It is in this generalized sense that we will treat the soliton in this work.

Recently the emphasis in the investigations of nonlinear waves has been more and more shifted to three-dimensional soliton-like formations that can be treated as a symbiosis of spatial and temporal solitons, which requires consideration of nonlinearity and dispersion of the medium and diffraction of the signal. Compensation for nonlinear self-compression in all directions by dispersion and diffraction leads to formation of light bullets, that is, stable light energy bunches localized in all directions [2]. Exact mathematical methods are rarely successful in describing these objects. Therefore, hopes are pinned on approximate approaches.

An approach to description of self-focusing of optical beams (spatial solitons) is proposed in [1]. The investigation within this approach is reduced to solving equations of ideal fluid dynamics. The roles of the fluid density and the velocity potential are played by the beam intensity and eikonal respectively. In [3] this approach is generalized to temporal solitons. It is based on the averaged Lagrangian (AL) method [4], which is used to derive hydrodynamic-type equations for soliton parameters.

In this work we investigate dynamics of axially symmetrical and vortex light beams in gradient glass fiber and of spatial-temporal solitons in the second harmonic generation mode. In both cases the investigation will be reduced to the analysis of hydrodynamic equations.

2. LIGHT BEAMS IN GRADIENT GLASS FIBER

Let a long monochromatic light beam with a frequency ω propagate along the z axis in gradient glass fiber whose refractive index is transversely inhomogeneous. Let, in addition, the glass fiber have cubic nonlinearity characterized by the third-order nonlinear susceptibility $\chi^{(3)}$. Then the envelope ψ of the electric field of the beam obeys the equation

$$i\frac{\partial\psi}{\partial z} - q(\mathbf{r})\psi - \alpha|\psi|^2\psi = \frac{c}{2n_0\omega}\Delta_\perp\psi. \qquad (1)$$

Here the second left-hand side term allows for the dependence of the linear susceptibility $\chi = \chi_0 + \widetilde{\chi}(\mathbf{r})$ on the transverse coordinates given by the vector $\mathbf{r}$; χ_0 is the constant component of the susceptibility, and $\widetilde{\chi}(\mathbf{r})$ has a property $\widetilde{\chi}(\mathbf{r}) = 0$ on the assumption that the origin of the transverse coordinates is at the center of the glass fiber cross

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section. Then $q(\mathbf{r})=2\pi\omega\widetilde{\chi}(\mathbf{r})/cn_0$, $\alpha=6\pi\chi_3\omega/cn_0$, $n_0=\sqrt{1+4\pi\chi_0}$ is the homogeneous part of the refractive index, χ_3 is the nonlinear susceptibility of the third order, c is the speed of light in the vacuum, and $\Delta_\perp$ is the transverse Laplacian.

To continue, we put

$$\psi = \sqrt{\rho}\exp\Big(-i\frac{n_0\omega}{c}\Phi\Big), \tag{2}$$

where ρ and Φ are the real functions of the coordinates.

Substituting (2) into (1) and separating the real part from the imaginary part, we arrive at the system

$$\frac{\partial\rho}{\partial z} + \nabla_\perp(\rho\nabla_\perp\Phi) = 0, \tag{3}$$

$$\frac{\partial\Phi}{\partial z} + \frac{(\nabla_\perp\Phi)^2}{2} - \frac{c}{n_0\omega}q(\mathbf{r}) + f(\rho) = \gamma\Big(\frac{c}{n_0\omega}\Big)^2\frac{(\Delta_\perp\sqrt{\rho})}{\sqrt{\rho}}, \tag{4}$$

where $\nabla_\perp$ is the transverse gradient, $\gamma=1/2$, and

$$f(\rho) = -\frac{c\alpha}{n_0\omega}\rho. \tag{5}$$

When the right-hand side is absent in (4), the system (3)–(4) is formally similar to the system of equations of two-dimensional ideal fluid dynamics [5]. In this case, (3) is the continuity equation and (4) is the Cauchy integral. The longitudinal z axis is the time and the eikonal Φ is the velocity potential; it is evident form (2) that the "fluid density" $\rho=|\psi|^2$ is positive and proportional to the light beam intensity.

The third left-hand side term in (4), which arises from the inhomogeneity of the linear susceptibility of the glass fiber, plays the role of the potential energy of the external field in which the fluid flows. Let $q(\mathbf{r})$ (and $\chi(\mathbf{r})$ as well) have a local minimum. Then the potential energy density $u(\mathbf{r})$ has a maximum here. Obviously, the imaginary fluid will spread beyond this region of unstable equilibrium, which will result in decreasing ρ. Consequently, the light beam energy will decrease here. In the case of the local $q(\mathbf{r})$ maximum, by similar reasoning we come to the conclusion that beam energy will accumulate in this region. The analyzed effect of the third left-hand side term in (4) corresponds to the linear refraction of the light beam and is strictly consistent with the Fermat principle.

The fourth term in this equation is related to the internal fluid pressure p and has the form $\int dp/\rho$ [5]. Equating this expression and the fourth left-hand side term in (4) and differentiating with respect to ρ with allowance for (5), we obtain the flow equation

$$\frac{dp}{d\rho} = -\frac{c\alpha}{n_0\omega}\rho. \tag{6}$$

It is evident that $dp/d\rho>0$ (normal fluid) if $\alpha<0$, which is equivalent to $\chi_3<0$. In this case, if the density ρ increases in any local area, the pressure in this area increases as well. As a result, this inhomogeneity disappears because the fluid tends to leave this area for regions with lower pressure. Returning to our optical beam, we state that it undergoes defocusing. Now let $\alpha\sim\chi_3>0$. Then, as is evident from (6), $dp/d\rho<0$ (Chaplygin gas [6]), which corresponds to self-focusing of the beam.

The right-hand side in (4) is beyond the optical-hydrodynamic analogy and describes the effect of diffraction on the beam propagation. Consequently, the neglect of the right-hand side in (4) corresponds to the geometrical optics approximation.

After this qualitative consideration we proceed to analyze some of the explicit solutions to the system (3)–(4). The following expressions obey equation (3):

$$\rho = \rho_0\frac{R_0^2}{R^2(z)}F\left[\frac{r}{R(z)}\right], \qquad \Phi = f_1(z) + \frac{r^2}{2}\frac{R'(z)}{R(z)} + \frac{c}{n_0\omega}m\varphi. \tag{7}$$

Here r is the radial component of the cylindrical coordinate system, which has a sense of the distance from the focal z axis to the observation point, φ is the azimuthal angle, m is an integer, R is the z-dependent characteristic transverse radius (aperture) of the beam, R_0 and ρ_0 are, respectively, the radius and the amplitude squared of the beam at the entrance to the medium at $z=0$, and F is the function determined after substitution of (7) into (4).

It is evident from (2) and the second expression (7) that the condition of periodicity of ψ in φ is automatically satisfied. Note that the last term in (7) with $m=1$ arises in hydrodynamics in lift force calculations [5].

We begin with the axially symmetrical case corresponding to $m=0$ [1, 7, 8]. Considering that

$$\tilde{\chi} = \eta r^2/l^2, \tag{8}$$

where l is the characteristic inhomogeneity scale of the linear susceptibility and $\eta=\pm1$, and substituting (7) into (4), we obtain

$$f_1' + \frac{r^2}{2}\frac{R''}{R} - \frac{2\pi}{n_0^2}\eta\frac{r^2}{l^2} - \frac{c\alpha}{n_0\omega}\rho_0\frac{R_0^2}{R^2}F = \Big(\frac{c}{n_0\omega}\Big)^2\frac{(\Delta_\perp\sqrt{F})}{2\sqrt{F}}. \tag{9}$$

Obviously, $\eta=+1$ corresponds to the defocusing linear refraction and $\eta=-1$ to the focusing one.

First, we solve the problem within the geometrical optics approximation, setting the right-hand side in

(9) equal to zero. Then it is convenient to choose the function $F(r/R)$ in the form

$$F = 1 - \frac{r^2}{R^2}, \quad 0 \leqslant r \leqslant R. \tag{10}$$

This expression corresponds to the beam with the intensity monotonically decreasing from the maximum (at the center) to zero (at $r = R$).

Substituting (10) into (9) and separately setting the coefficients of r to the power 0 and 2 equal to zero while ignoring the right-hand side, we obtain

$$f_1' = \frac{b}{R^2}, \tag{11}$$

$$R'' - \mu R + \frac{a}{R^3} = 0, \tag{12}$$

where $a = (2c\alpha/n_0\omega)\rho_0 R_0^2$, $\mu = (4\pi/n_0^2 l^2)\eta$, and $b = a/2$.

Equation (12) is an analogue of Newton's second law for a particle of unit mass with the coordinate R, where z plays the role of time. To this motion there corresponds the "potential energy" of the form

$$U = -\frac{\mu}{2}R^2 - \frac{a}{2R^2}. \tag{13}$$

Obviously, $U(R)$ has a stable minimum only for $\mu < 0$ and $a < 0$, which corresponds to the focusing linear refraction and defocusing self-refraction.

The ratio R'/R has the sense of the curvature of the light beam wave fronts. Assuming that wave fronts are plane at the entrance to the medium, we have $R'(0) = 0$. Then equation (12) for $\mu < 0$ and $a < 0$ has a spatially oscillating solution

$$R = \left[R_0^2 + (R_1^2 + R_0^2)\sin\left(\sqrt{-\mu}z\right)\right]^{1/2}, \tag{14}$$

where $R_1^2 = a/\mu R_0^2$.

Thus, the beam aperture experiences oscillations, taking on the values between R_0 and R_1. The phase velocity and the wave front curvature have an oscillating character (see (2), (7), (11), (19)). The period of the oscillations depends on the inhomogeneity of the glass fiber and is $\sqrt{\pi}n_0 l$.

It is quite clear that in the case of the defocusing linear refraction and focusing self-refraction the beam undergoes self-focusing at $3\chi_3\rho_0 > (R_0/l)^2$. Otherwise, unlimited defocusing occurs.

To consider diffraction, note that expression (10) corresponds to the beam with sharp transverse boundaries at $r = R$. Diffraction should favor smearing of the boundaries. Based on (10), we now choose F in the form

$$F = \exp\left(-\frac{r^2}{R^2}\right). \tag{15}$$

Substituting (15) into (9), we obtain

$$f_1' + \frac{r^2}{2}\frac{R''}{R} - \frac{2\pi}{n_0^2}\eta\frac{r^2}{l^2} - \frac{c\alpha}{n\omega}\rho_0\frac{R_0^2}{R^2}\exp\left(-\frac{r^2}{R^2}\right) = \left(\frac{c}{n_0\omega}\right)^2\left(\frac{r^2}{2R^4} - \frac{1}{R^2}\right). \tag{16}$$

Now we use the near-axis approximation [1, 9], examining regions adjacent to the center of the beam, where the intensity is at maximum and $r^2/R^2 \ll 1$. Putting $\exp(-r^2/R^2) \approx 1 - r^2/R^2$ on the left-hand side of (16) and equating the coefficients of r to the power 0 and 2 on both sides, we arrive at equations (11) and (12) where

$$a = \frac{2c\alpha}{n_0\omega}\rho_0 R_0^2 - \left(\frac{c}{n_0\omega}\right)^2,$$

$$b = \frac{c\alpha}{n_0\omega}\rho_0 R_0^2 - \left(\frac{c}{n_0\omega}\right)^2,$$

and the expression for μ remains unchanged.

It is seen that a can be negative in the case of focusing nonlinearity ($\alpha > 0$) as well. This requires fulfillment of the condition

$$\rho_0 R_0^2 < \frac{\lambda^2}{48\pi^3\chi_3}, \tag{17}$$

where $\lambda = 2\pi c/\omega$ is the wavelength of the optical beam.

Inequality (17) corresponds to the known condition for the beam threshold power, the excess of which leads to the self-focusing of the beam [1].

Now we turn to considering vortex solutions for the light beam when $m \neq 0$ in (12), confining ourselves to the geometrical optics approximation. Substituting (7) into (4), we obtain

$$f_1' + \frac{r^2}{2}\frac{R''}{R} + \left(\frac{c}{n_0\omega}\right)^2\frac{m^2}{2r^2} - \frac{2\pi}{n_0^2}\eta\frac{r^2}{l^2} - \frac{c\alpha}{n_0\omega}\rho_0\frac{R_0^2}{R^2}F = 0. \tag{18}$$

To satisfy this equality, F should be taken in the form

$$F = 1 - g\frac{r^2}{R^2} - h\frac{R^2}{r^2}, \tag{19}$$

where g and h are constants.

Substituting (19) into (18), we find that

$$h = -\frac{cm^2}{2n_0\omega\alpha\rho_0 R_0^2}, \tag{20}$$

and f_1 and F obey equations (11) and (12), respectively, with the coefficient values given immediately behind them.

The function F defined in accordance with (19) has two real roots $r_2 > r_1$ only if g and h are positive,

$$r_{1,2} = R\left[\frac{1}{2g}\left(1 \mp \sqrt{1-4gh}\right)\right]^{1/2}. \tag{21}$$

Now the range of definition of F (and consequently ρ as well) is $r_1 \leqslant r \leqslant r_2$, where it has the maximum value $F_{\max} = 1 - 2\sqrt{gh}$ at $r = r_{\mathrm{m}} = (h/g)^{1/4}R$.

Since $h > 0$, then $\alpha \sim \chi_3 < 0$, as follows from (20). Thus, the nonlinearity must be defocusing to form a localized vortex in the gradient fiber glass. At the same time, it follows from (21) that

$$4gh < 1. \tag{22}$$

The periodic dependence $R(z)$ is described by (15). It is evident from (21) that the interval $r_2 - r_1$ defining the vortex existence region becomes narrower as R decreases. Since $F_{\max}$ remains unchanged, the amplitude of the vortex intensity increases according to the first expression in (7). When R increases, the reverse situation occurs.

3. SPATIAL-TEMPORAL SOLITONS

Now we go over from continuous beams (spatial solitons) to solitary pulses in nonlinear dispersion media, spatial-temporal solitons, considering their dynamics with allowance for transverse effects. To this end, we use the AL method [3, 4, 10], which we mentioned in the Introduction. Within this approach, an exact one-dimensional soliton solution is first found for the wave equation under consideration. Next, to allow for the effect of the transverse spatial measurements, an assumption is made that some of the parameters in these solutions depend on coordinates. The result is the so-called trial solutions. They are substituted into the Lagrangian that corresponds to the initial wave equation involving derivatives with respect to transverse coordinates. Then the resulting expression is averaged over time. Finally, the AL involving dependence on the variable parameters is obtained. Applying this averaged Lagrangian to the Euler–Lagrange equations for the variable parameters, we arrive at a system like (3) and (4).

Let us use the soliton mode of the second harmonic generation to demonstrate this procedure. If the phase and group matching conditions are fulfilled, the system of equations for the envelopes of the first ψ_1 and second ψ_2 harmonics has the form [11]

$$i\frac{\partial\psi_1}{\partial z} + \frac{k_2^{(1)}}{2}\frac{\partial^2\psi_1}{\partial\tau^2} - d_1\psi_1^*\psi_2 = \frac{c}{2n_0\omega}\Delta_\perp\psi_1, \tag{23}$$

$$i\frac{\partial\psi_2}{\partial z} + \frac{k_2^{(2)}}{2}\frac{\partial^2\psi_2}{\partial\tau^2} - d_2\psi_1^2 = \frac{c}{4n_0\omega}\Delta_\perp\psi_2. \tag{24}$$

Here $\tau = t - z/v_{\mathrm{g}}$ is the local time; v_{g} and n_0 are the linear group velocity and the refractive index, respectively, identical for both harmonics; d_1 and d_2 are the quadratic nonlinearity coefficients proportional to the second-order nonlinear susceptibility $\chi^{(2)}$ for both harmonics; and $k_2^{(1)}$ and $k_2^{(2)}$ are the coefficients of the group velocity dispersion (GVD) at the first and second harmonics, respectively.

If GVD coefficients are connected by the relation

$$k_2^{(2)} = 2k_2^{(1)}, \tag{25}$$

system (23)–(24) has temporal ($\Delta_\perp = 0$) soliton solutions of the form [12]

$$\psi_1 = \pm\frac{3k_2}{4\tau_{\mathrm{p}}^2}\sqrt{\frac{2}{d_1d_2}}\exp\left(i\frac{k_2z}{2\tau_{\mathrm{p}}^2}\right)\mathrm{sech}^2\left(\frac{\tau}{2\tau_{\mathrm{p}}}\right), \tag{26}$$

$$\psi_2 = -\frac{3k_2}{4\tau_{\mathrm{p}}^2d_1}\exp\left(i\frac{k_2z}{\tau_{\mathrm{p}}^2}\right)\mathrm{sech}^2\left(\frac{\tau}{2\tau_{\mathrm{p}}}\right). \tag{27}$$

From here on $k_2 \equiv k_2^{(1)}$ and τ_{p} is the duration of the pulses of both harmonics.

Now let us consider the transverse dynamics determined by the right-hand sides in (23) and (24). Performing the scaling transformation

$$\psi_1 = \sqrt{\frac{d_1}{2d_2}}\Phi_1, \quad \psi_2 = \Phi_2, \tag{28}$$

we write the density of the Lagrangian corresponding to system (23) and (24) as

$$L = L_1 + L_2 + L_{\mathrm{int}}, \tag{29}$$

where

$$L_1 = \frac{i}{2}\left(\Phi_1^*\frac{\partial\Phi_1}{\partial z} - \Phi_1\frac{\partial\Phi_1^*}{\partial z}\right) - \frac{k_2}{2}\left|\frac{\partial\Phi_1}{\partial\tau}\right|^2 + \frac{c}{2n_0\omega}|\nabla_\perp\Phi_1|^2, \tag{30}$$

$$L_2 = \frac{i}{2}\left(\Phi_2^*\frac{\partial\Phi_2}{\partial z} - \Phi_2\frac{\partial\Phi_2^*}{\partial z}\right) - k_2\left|\frac{\partial\Phi_2}{\partial\tau}\right|^2 + \frac{c}{4n_0\omega}|\nabla_\perp\Phi_2|^2, \tag{31}$$

$$L_{\mathrm{int}} = -\frac{d_1}{2}(\Phi_1^{*2}\Phi_2 + \Phi_1^2\Phi_2^*). \tag{32}$$

In agreement with (26)–(28), we choose the trial solutions in the form

$$\Phi_1 = \pm\frac{6k_2}{d_1}\rho^{2/3}\exp\left(-i\frac{n_0\omega}{c}\Phi\right)\mathrm{sech}^2(\rho^{1/3}\tau), \tag{33}$$

$$\Phi_2 = -\frac{3k_2}{d_1}\rho^{2/3}\exp\left(-2i\frac{n_0\omega}{c}\Phi\right)\mathrm{sech}^2(\rho^{1/3}\tau). \tag{34}$$

Here ρ and Φ are unknown functions of coordinates.

On substituting (33) and (34) in (29)–(32) and integrating over τ, we have

$$\int_{-\infty}^{+\infty} L\,dt = 216\left(\frac{k_2}{\gamma}\right)^2 \Lambda,$$

where the averaged Lagrangian is

$$\Lambda = \frac{\rho}{3}\left[\frac{\partial \Phi}{\partial z} + \frac{(\nabla_\perp \Phi)^2}{2}\right] + \frac{2ck_2}{5n_0\omega}\rho^{5/3} + \left(1+\frac{\pi^2}{30}\right)\left(\frac{c}{n_0\omega}\right)^2 \frac{(\nabla_\perp \rho)^2}{36\rho}. \tag{35}$$

Writing the corresponding Euler–Lagrange equations for Φ and ρ, we again arrive at the hydrodynamic-type system like (3), (4), where $q(\mathbf{r})=0$, $\gamma=(1+\pi^2/30)/3$, and

$$f(\rho) = \frac{2c}{n_0\omega} k_2 \rho^{2/3}. \tag{36}$$

Thus, on obtaining similar systems of the hydrodynamic type in dealing with spatial and spatial-temporal solitons, we can investigate them on a unified basis. For example, the qualitative analysis (like the one performed in the previous section) shows that soliton (33) and (34) propagates in the self-focusing mode when $k_2<0$ and in the defocusing mode when $k_2>0$.

Let us find localized axially symmetrical non-vortex solutions to the system of (3), (4), (36), putting $m=0$ in (7). It is clear that within the geometrical optics approximation we have $F=(1-r^2/R^2)^{3/2}$ here. Then, using the right-hand side in (4) to allow for diffraction, we set $F=\exp(-3r^2/2R^2)$ by analogy with the previous section. It is evident from (7), (33), and (34) that this approximation corresponds to that the envelopes of both harmonics decrease with increasing r as $\sim \exp(-r^2/R^2)$. As a result, we arrive at the equations

$$f' = -\frac{2ck_2}{n_0\omega}\rho_0^{2/3}\frac{R_0^{4/3}}{R^{4/3}} - 3\gamma\left(\frac{c}{n_0\omega}\right)^2 \frac{1}{R^2}, \tag{37}$$

$$R'' = -\frac{4ck_2}{n_0\omega}\rho_0^{2/3}\frac{R_0^{4/3}}{R^{7/3}} + \frac{9}{2}\gamma\left(\frac{c}{n_0\omega}\right)^2 \frac{1}{R^3}. \tag{38}$$

To equation (38) there corresponds the "potential energy" of the form

$$U = \frac{3ck_2}{n_0\omega}\rho_0^{2/3}\frac{R_0^{4/3}}{R^{4/3}} + \frac{9}{4}\gamma\left(\frac{c}{n_0\omega}\right)^2 \frac{1}{R^2}. \tag{39}$$

Hence it follows that $U(R)$ has a minimum only if $k_2<0$. As is pointed out above, this corresponds to the self-focusing mode at the eikonal stage. Under particular conditions, diffraction can further stop this process and lead to formation of a two-component light bullet. Let us determine these conditions. To the minimum in $U(R)$ there corresponds the bullet radius

$$R_{\mathrm{m}} = 0.295\left(\frac{c}{n_0\omega|k_2|}\right)^{3/2}\frac{\tau_0^3}{R_0^2}, \tag{40}$$

where $\tau_0=\rho_0^{-1/3}$ is the duration of the bullet at its transverse center.

Thus, while propagating, the bullet pulses in such a way that its transverse size varies relative to R_{m}. Its amplitude, duration, and phase velocity also vary. Its wave fronts periodically bend. Putting $R_{\mathrm{m}}=R_0$ in (40), we obtain

$$R_0 = 0.706\sqrt{\frac{c}{n_0\omega|k_2|}}\,\tau_0. \tag{41}$$

No pulsing occurs, and the transverse size of the bullet is proportional to its duration.

If $U>0$, the soliton irreversibly expands. Consequently, the condition $U<0$ must be fulfilled to ensure stability of the bullet. Then, setting $R=R_0$ in (39), we have

$$R_0 > 0.576\sqrt{\frac{c}{n_0\omega|k_2|}}\,\tau_0. \tag{42}$$

Substituting (42) into (42), we obtain the upper estimate of the equilibrium radius

$$R_{\mathrm{m}} < 1.06\sqrt{\frac{c}{n_0\omega|k_2|}}\,\tau_0. \tag{43}$$

Apart from the phase and group matching conditions, equality (25) is important for correctness of the results obtained in this section. All these conditions can hardly be strictly fulfilled at the same time, but it can be done approximately if the frequencies ω and 2ω are below the characteristic frequencies of resonant absorption, which is true, for example, for the terahertz range. In this case the wave number $k\approx n_0\omega/c + s\omega^3$, where s is a constant and the second term is small as compared with the first one. Then the phase and group velocities of both harmonics can be approximately taken to be c/n_0. At the same time, $k_2\equiv \partial^2 k/\partial\omega^2 = 6s\omega$ and (25) hold automatically. However, in an equilibrium medium featuring only time dispersion we have $s>0$. Consequently, $k_2>0$, while formation of the above bullets requires that $k_2<0$. In a mcirodispersed (granulated) medium, where space dispersion is of importance, s can become negative under particular conditions [13], which results in that $k_2<0$ and (25) is simultaneously fulfilled. Then formation of the above bullets becomes possible.

Note that light bullets in quadratically nonlinear media were earlier predicted using, among others, the variational approach (see [14] and references therein). However, this work is the first to reduce the investigation to hydrodynamic-type equations (3) and (4) and seek their solutions in the form of (7).

4. CONCLUSIONS

Summing up, we note that the AL method leads to equations of the hydrodynamic type for solitons of a large number of equations [10, 15]. The corresponding equations differ by functions $f(\rho)$ and right-hand sides of Cauchy integral (4).

It is worth mentioning that the AL method is valid if the temporal soliton is formed before the transverse dynamics effects (self-focusing, defocusing, etc.) come into play. The nonlinear refraction, which is present in the geometrical optics approximation, manifests itself earlier than the wave properties of the soliton that correspond to diffraction. If we introduce the lengths of dispersion l_{d}, refraction l_{r}, and diffraction l_{D}, all that is said above will be written as $l_{\mathrm{d}} \ll l_{\mathrm{r}} \ll l_{\mathrm{D}}$. It turns out that to meet this condition is not so difficult if we consider the automatically arising equality $l_{\mathrm{r}} = \sqrt{l_{\mathrm{d}} l_{\mathrm{D}}}$ [7].

Thus, the above approach stemming from the theory of nonlinear light beams [2] gives a rather full and qualitatively correct description of the dynamics of spatial-temporal solitons.

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