

NONLINEAR DYNAMICS OF THE INTERACTION OF NATIONAL INCOME, INTEREST RATE, AND PRICE LEVEL

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We study the dynamics of a macroeconomic system formed by the interacting national income, interest rate, and price level. This dynamics can be described by a discrete system in the three-dimensional space. We study this system and show that there is a curve formed by fixed points that give equilibria on the market of money, commodities, and services. We prove that there exists a foliation transverse to the curve each layer of which is an integral manifold for the system. In the direction transverse to the layers, the dynamics changes from the equilibrium state to chaos. The paper contains both the theoretical results and the data of numerical experiments. We also consider perturbations of the system by small random uncontrolled actions simulating, in a natural way, the environmental influence on the system.

1. Introduction

We consider the dynamics of the macroeconomic “national income–interest rate–price level” system. This dynamics is described by the “IS-LM”-model, basic for the description of the contemporary market economy [2, 5, 8]. To simulate the macroeconomic dynamics, we use the discrete dynamical system

$$u_{n+1} = F(u_n), \quad (1)$$

where u_n describes the state of economics at time $t = n$, $n \in \mathbb{Z}$. In the monograph [1, p. 283], a discrete dynamical system of the form (1) is used to simulate the interaction of the above-mentioned parameters of the economic system. This system has the form

$$\begin{aligned} x_{n+1} &= x_n \exp(a(1 - x_n y_n^{m/s} z_n)), \\ y_{n+1} &= y_n \exp(b(1 - x_n y_n^{m/s} z_n)), \\ z_{n+1} &= z_n \exp(c(y_n - z_n)), \end{aligned} \quad (2)$$

where $x = P/P_e$, P is a price level, P_e is the equilibrium value of the price level, $y = (r_e/r)^s$, r is a rate interest, r_e is the equilibrium value of the rate interest, $z = Y/Y_e$, Y is a national income, and Y_e is the equilibrium value of the initial income. All variables are dimensionless positive quantities and their variations are regarded as deviations from the equilibrium state. The parameters a , b , c , m , and s , $0 < s < 1$, are positive. We can say that the variable x is proportional to the price level, the variable z is proportional to the national income, and the variable y is inversely proportional to the s th power of the interest rate. In [1, p. 285], one can find the results of numerical experiments and the presence of periodic orbits is demonstrated.

The aim of the present paper is the detailed investigation of the introduced discrete system and the description of the qualitative picture of its global dynamics. The presented investigation is mathematically rigorous. We obtain

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general theoretical results and perform numerical calculations for specific parameters of the system. First, we study a simple case $m = 0$. The equality $m = 0$ means that the demand is independent of the interest rate. It is shown that the system have a curve K formed by fixed points. A foliation with invariant layers lies transversely with respect to the curve K . This foliation is defined as level surfaces of the function

$$U = \frac{x^b}{y^a},$$

i.e., the surface has the form

$$W = \left\{ (x, y, z) : \frac{x^b}{y^a} = \text{const} \right\}.$$

A bifurcation of the topological structure of orbits of the system occurs from layer to layer. In this case, we have layers with stable equilibrium state and, at the same time, layers in which almost every orbit starting near a fixed point approaches the chaotic mode. In Sec. 4, it is shown that a similar topological structure exists for any $m > 0$. Moreover, we prove that, for any $m \geq 0$, system (2) is topologically equivalent to a system with $m = 0$ and a different parameter a .

Action of the Environment. The environment in which we study the development of macroeconomics affects the system. An external action can be regarded as an uncontrolled perturbation. To simulate external actions, we assume that the perturbation randomly depends on time n and that it is small and can be added to the result as a whole. Thus, we obtain equations of the form

$$\begin{aligned} x_{n+1} &= x_n \exp(a(1 - x_n y_n^{m/s} z_n)) + \varepsilon q_1(n), \\ y_{n+1} &= y_n \exp(b(1 - x_n y_n^{m/s} z_n)) + \varepsilon q_2(n), \\ z_{n+1} &= z_n \exp(c(y_n - z_n)) + \varepsilon q_3(n), \end{aligned} \tag{3}$$

where ε is a small positive number, $q_i(n)$ takes random values from the segment $[-1, 1]$ and chaotically depends on time n . To simulate the perturbation $q_i(n)$, we use the dependence

$$q(n+1) = 1 - 2q^2(n), \quad q \in [-1, 1],$$

where the initial value $q(0)$ is specified for each $i = 1, 2, 3$. It is known [3, p. 115; 13, p. 60] that, for almost all (with respect to the Lebesgue measure) initial values $q(0)$, the orbit $\{q_n\}$ is chaotic and distributed over the segment $[-1, 1]$ with the density

$$\rho = \frac{1}{\pi(1-x^2)^{1/2}}.$$

Software. The numerical analyses were performed on the basis of algorithms developed by the author and substantiated in [4, p. 204; 11, p. 188]. The software for these algorithms and visualizations were realized by M. Sen'kov, a graduate of the St.-Petersburg University.

2. Dynamics Near Fixed Points

Consider the discrete dynamical system (2). As already indicated, we first study the case $m = 0$, which somewhat simplifies calculations and illustrates the principal specific features of the dynamics. We also use this assumption in the next sections. The balance state is specified by the fixed points of system (1), i.e., by the equation

$$F(u) = u.$$

Passing to the coordinates (x, y, z) and system (2), we get the equalities $xy - 1 = 0$ and $y - z = 0$. Thus, the fixed points of the system fill the curve $K = \{xy = 1, y = z\}$. The projection of K onto the (XY) -plane is the hyperbola $xy = 1$ and the projection of K onto the (YZ) -plane is the straight line $y = z$. The topological type of the fixed point $u^* \in K$ of system (1) is specified by the multipliers or eigenvalues of the differential $DF(u^*)$. The invariance of the curve K implies that the vector tangential to K is an eigenvector of the differential. Moreover, since the mapping of F onto K is identical, we conclude that $\lambda = 1$ is the eigenvalue of the differential $DF|_K$. On the curve K , the differential of the right-hand side of system (2) has the form

$$D = \begin{pmatrix} 1-a & 0 & -ax^2 \\ -by^2 & 1 & -b \\ 0 & cz & 1-cz \end{pmatrix}. \quad (4)$$

The multipliers of fixed points are given by the equation

$$\det(D - \lambda E) = (1 - \lambda)(\lambda^2 - \lambda(2 - a - cy) + 1 - a + (b + a - 1)cy).$$

It is clear that the multiplier $\lambda = 1$ corresponds to the curve of fixed points K . The other multipliers $\lambda_{1,2}$ are given by the equation

$$\lambda^2 - \lambda(2 - a - cy) + 1 - a + (b + a - 1)cy = 0. \quad (5)$$

This yields

$$\lambda_{1,2} = \frac{2 - a - cy \pm \sqrt{\Delta}}{2},$$

where the discriminant

$$\Delta = c^2y^2 - 2cy(a + 2b) + a^2.$$

It is easy to see that $\lambda_{1,2} \neq 1$ for $y > 0$. This implies that the proper subspace for $\lambda_{1,2}$ is transverse to K . If $|\lambda_{1,2}| < 1$, then the differential contracts to K . At the same time, for $|\lambda_{1,2}| > 1$, the differential stretches from K . If $|\lambda_1| < 1$ and $|\lambda_2| > 1$, then we have the hyperbolic dynamics near K . The discriminant Δ is negative if y lies between the roots y_1 and y_2 of the equation

$$c^2y^2 - 2cy(a + 2b) + a^2 = 0, \quad (6)$$

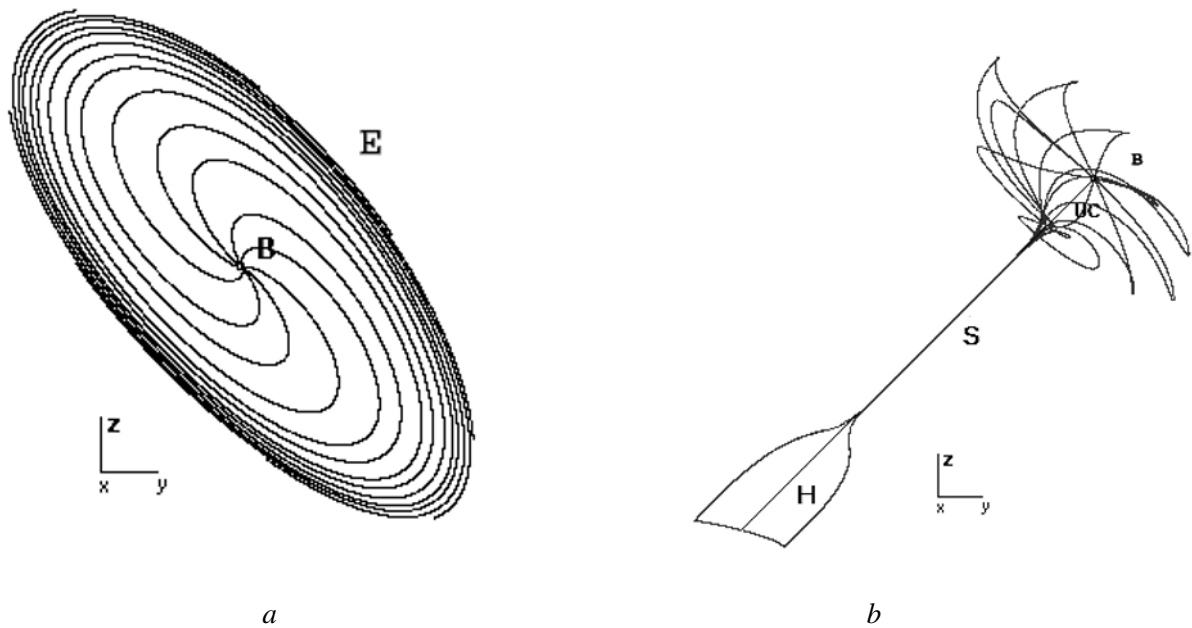


Fig. 1. Dynamics near the curve K of fixed points of system (2).

where

$$y_{1,2} = \frac{a + 2b \pm 2\sqrt{b(a+b)}}{c}.$$

Thus, for $a = 2.4$, $b = 0.87$, and $c = 0.9$, we have $y_1 = 0.852$ and $y_2 = 7.51$. Hence, for $0.852 < y < 7.51$, the multipliers $\lambda_{1,2}$ are complex conjugate. In this case, the free term in Eq. (5) is the squared modulus of the multiplier

$$1 - a + (b + a - 1)cy = \lambda_1 \lambda_2 = |\lambda|^2,$$

which enables us to determine the domains of stable and unstable fixed points. For $a = 2.4$, $b = 0.87$, $c = 0.9$, and $y = 0.852$, we have $|\lambda|^2 = 0.3406$, i.e., we have a stable focus in the plane transverse to the curve of fixed points. Numerical experiments show that, for $1 - a + (b + a - 1)cy = 1$, a Neimark–Sacker bifurcation occurs: the equilibrium state loses the stability and a stable invariant curve homeomorphic to a circle separates from it. For $a = 2.4$, $b = 0.87$, and $c = 0.9$, a bifurcation occurs for $y = 1.174743$. On the curve K , stable equilibrium states lie between the points $0.852 < y < 1.174743$, and, for $y > 1.174743$, equilibrium states are unstable. By the Pliss theorem [6, p. 1297; 7, p. 1139; 12, p. 939], an invariant disk lies above each fixed point near the curve K . This disk is a stable manifold W^s for $|\lambda| < 1$ ($0.852 < y < 1.174743$) or an unstable manifold W^u for $|\lambda| > 1$ ($1.174743 < y < 7.51$). The invariant disk W^u for the point B (0.851; 1.175; 1.175) is shown in Fig. 1a. The disk W^u contains the unstable equilibrium state B and the invariant stable ellipse E . The orbits start at the point B for $n = -\infty$ and end on S for $n = +\infty$.

In the curve K , the equilibrium states with two negative multipliers whose absolute values are less than 1 lie between the points $0.700 < y < 0.852$. For $0.685 < y < 0.700$, we have the hyperbolic case: the absolute value of one multiplier is greater than 1 and the absolute value of the other multiplier is smaller than 1. Near $y = 0.685$, one multiplier is equal to zero. This means that the Jacobian of the right-hand side of system (2) $\det D = 0$. For $y < 0.685$, one multiplier is positive and another is negative. In this case, the differential changes orientation (on unstable manifold). The sign of the Jacobian $\det D$ specifies whether the dynamical system preserves or does not

preserve its orientation at this point. Therefore, the equation $\det D(x, y, z) = 0$ describes the surface Π on which the system changes orientation. Thus, every equilibrium state in the plane transverse to K can be stable, unstable with complex multipliers, and hyperbolic. In the last case, one multiplier is negative and the second changes sign on Π . The segments $H = \{\text{hyperbolic fixed points}\}$, $S = \{\text{stable fixed points}\}$, and $UC = \{\text{unstable fixed points with complex multipliers}\}$ are shown in Fig. 1b.

3. Foliation with Invariant Layers

In this section, we show that there exists a function $U(x, y, z)$ whose level surfaces $U(x, y, z) = \text{const}$ are invariant for the discrete system

$$\begin{aligned}x_{n+1} &= x_n \exp(a(1 - x_n z_n)), \\y_{n+1} &= y_n \exp(b(1 - x_n z_n)), \\z_{n+1} &= z_n \exp(c(y_n - z_n)).\end{aligned}\tag{7}$$

The function $U(x, y, z)$ is an analog of the integral for an autonomous system of differential equations. To determine the function U , we note that the first and second equations of system (7) differ only by the values of a and b . Raising the first equation to the power b and the second to the power a , we obtain the same expression $\exp(ab(1 - x_n z_n))$. This enables us to construct the function U .

Proposition 1. *The level surfaces of the function*

$$U = \frac{x^b}{y^a}$$

are invariant for system (7).

Proof. The level surface H is given by the equation

$$U(x, y, z) = \text{const}.$$

The invariance of H means that if an orbit of system (7) starts on H , then it remains on H forever. In other words, the value of the function U does not vary from iteration to iteration. Thus, to prove the invariance, it is necessary to check the equality

$$U(x_n, y_n, z_n) = U(x_{n+1}, y_{n+1}, z_{n+1}).$$

Indeed,

$$U(x_{n+1}, y_{n+1}, z_{n+1}) = \frac{x_{n+1}^b}{y_{n+1}^a} = \frac{x_n^b \exp(ba(1 - x_n z_n))}{y_n^a \exp(ab(1 - x_n z_n))} = \frac{x_n^b}{y_n^a} = U(x_n, y_n, z_n).$$

The proposition is proved.

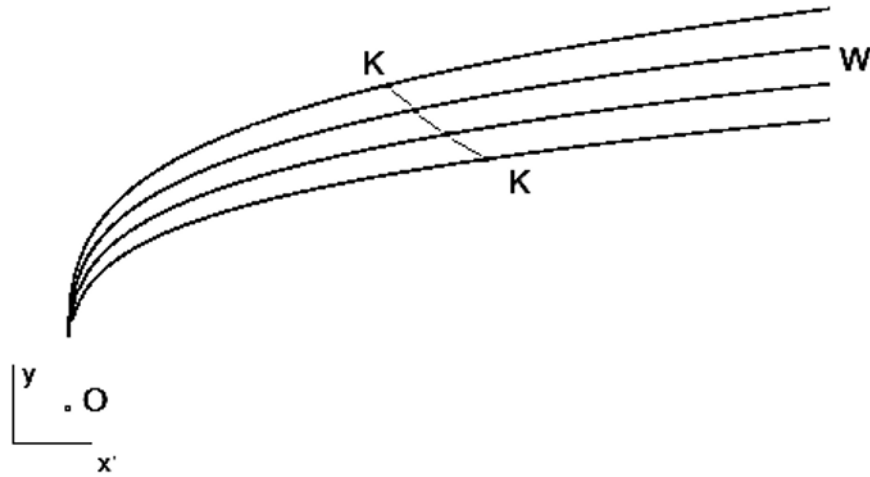


Fig. 2. Curve of fixed points $K = \{(x, y, z) : xy = 1, y = z\}$ and invariant layers $W = \{(x, y, z) : x = hy^{a/b}, h = \text{const}\}$.

Each surface

$$W = \{(x, y, z) : x = hy^{a/b}, h = \text{const},$$

is transverse to the curve of fixed points K (Fig. 2). In addition, as shown above, there exist invariant disks $W^{s,u}$ in the vicinity of K . In view of the invariance, the disks $W^{s,u}$ must lie in the layers of W . Thus, the invariant foliation generated by the stable W^s and unstable W^u disks exists not only near the curve K but it is globally defined. Since the function U does not explicitly depend on the variable z , the invariant layer $W = \{(x, y, z) : x = hy^{a/b}, h = \text{const},$ is a ruled surface with straight lines parallel to the Z -axis. The system on the surface $W = \{(x, y, z) : x = hy^{a/b}\}$ is defined in the form

$$y_{n+1} = y_n \exp(b(1 - hy_n^{a/b}z_n)), \quad (8)$$

$$z_{n+1} = z_n \exp(c(y_n - z_n)),$$

where the number $h > 0$ defines an invariant layer. For example, $h = 1$ defines a surface passing through the equilibrium state $(1, 1, 1)$. For an arbitrary surface $W(h)$, the equilibrium state is defined by the equalities

$$xy = 1, \quad y = z, \quad x = hy^{a/b}.$$

This implies that the equilibrium state has the coordinates

$$\left(h^{\frac{b}{a+b}}, h^{-\frac{b}{a+b}}, h^{-\frac{b}{a+b}}\right)$$

on each layer $W(h)$.

We study the changes in the dynamics of the system in the layers

$$W(h) = \{(x, y, z) : x = hy^{a/b}\}$$

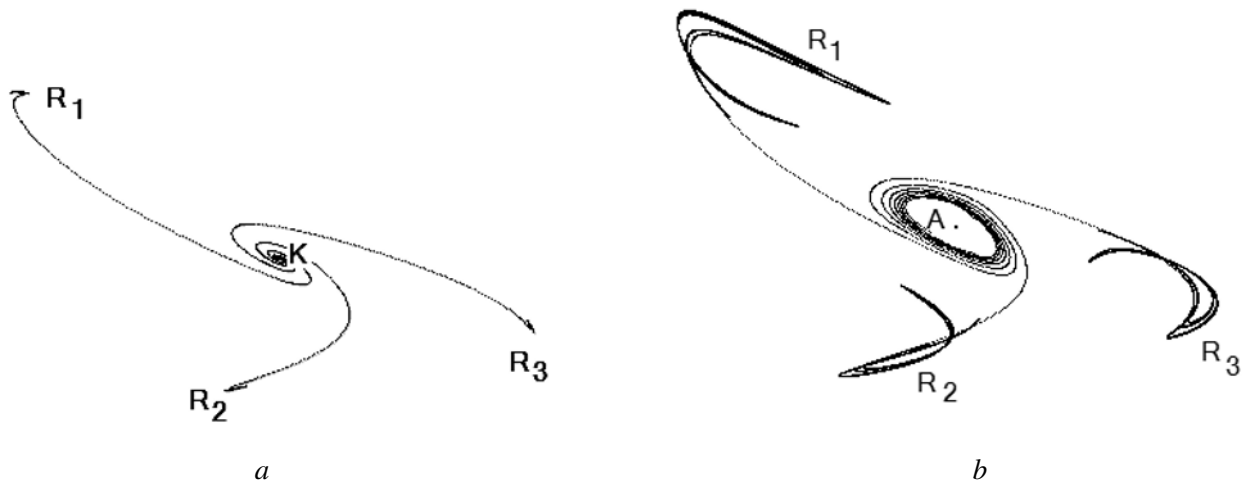


Fig. 3. Birth of a stable invariant ellipse with variation in h from 0.8 to 0.58.

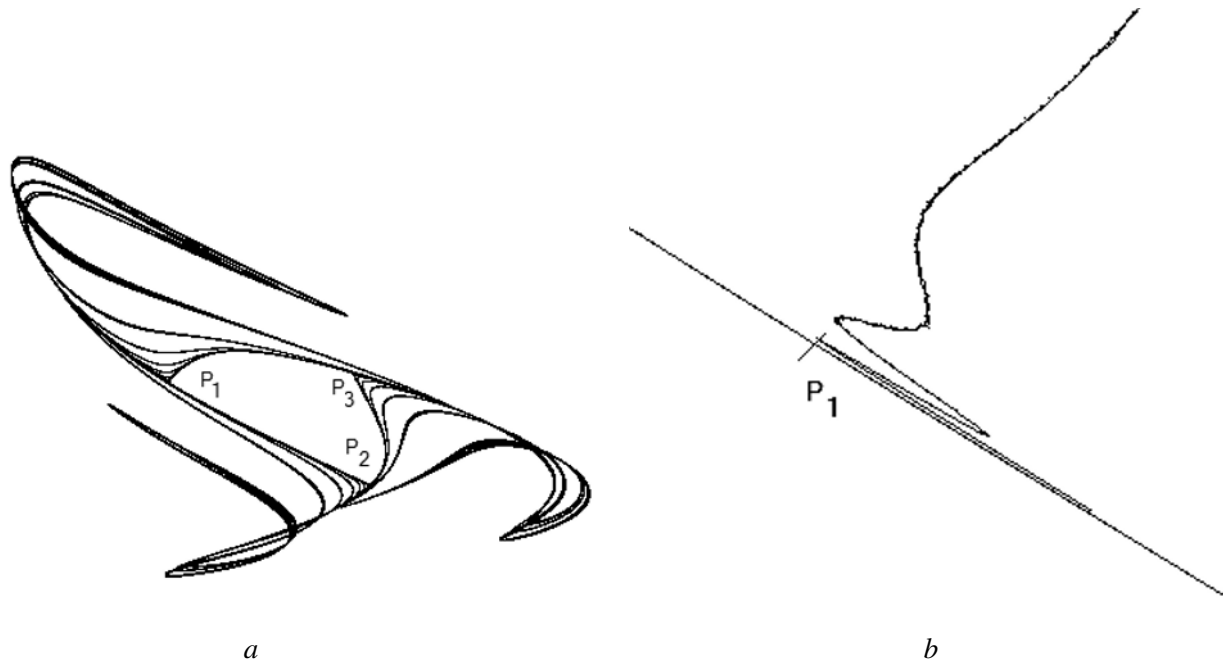


Fig. 4. Chaos in the invariant layer $W(h = 0.515970)$ and the closure of the unstable manifold $W^u(P)$. Intersection of the unstable and stable manifolds at the 3-periodic point P .

depending on the value of h . In other words, we consider bifurcations of system (8) caused by the variations of the parameter h . In each layer $W(h)$, there exists a fixed point $W(h) \cap K$ of system (8). For a fixed layer, we denote the fixed point by K^* . Consider system (8) for $a = 2.4$, $b = 0.9$, and $c = 0.9$. In the layer $W(h = 1)$, there is a stable equilibrium state $K^*(1, 1)$ with complex multipliers. As h decreases, the character of stability begins to vary. In the layer $W(h = 0.8)$ (see Fig. 3a), there is a 3-periodic invariant set $R = \{R_1, R_2, R_3\}$, and orbits goes from this set to the stable equilibrium state K^* . As the parameter h varies from 0.8 to 0.58, a stable invariant ellipse A is born from the stable equilibrium state and the equilibrium state K^* loses stability, i.e., a Neimark–Sacker bifurcation occurs (see Fig. 3b). In this case, the invariant set R increases. A subsequent

decrease in the parameter h is accompanied by the birth of the hyperbolic 3-periodic orbit P on the stable ellipse A (see Fig. 4a). In the case where h reaches the value 0.515970, the unstable manifold $W^u(P_3)$ transversely intersects the stable manifold $W^s(P_1)$ of the orbit P (0.8037147; 1.33638665) (see Fig. 4b). These manifolds are constructed and their transversality is estimated according to [4, p. 204; 11, p. 188]. By the Smale theorem [9, p. 351], a transverse intersection generates chaos near points of intersection. Moreover, the set A loses stability and coalesces with the set R by forming one invariant set Ω , which is the closure of the unstable manifold $W^u(P)$ of the orbit P (see Fig. 4a).

Note that an orbit starting near the equilibrium state K^* (1.1977742; 1.1977742) attains the set Ω and then wanders in it. The orbit of the point B (1.2; 1.2) is shown in Fig. 5a. The entropy E of the system on the invariant set Ω is estimated as the exponent of 2 of an increase in the length of curve [10, p. 1331]. We obtain the lower bound $E = 0.69314$. Since the entropy is the measure of chaos, we can state that the system admits chaos on the set Ω , which is clearly illustrated by the orbit of the point B in Fig. 5a.

A subsequent decrease in the parameter h leads to an increase in chaos. Numerical experiments show that system (2) has layers where chaos in attractors reaches large values. A chaotic set in the layer $W(h = 0.3)$ on a 1:10 scale is shown in Fig. 5b. This set is ω -limiting for any orbit starting near the fixed point K (1.3886835; 1.3886835).

Another picture of bifurcations occurs with increase in the parameter h . System (8) on the layer $W(h)$, $h > 0.8$, has a stable equilibrium state K up to $h = 4.02787$. Multipliers are, first, complex and then become real. For $h = 4.02787$, the equilibrium state K loses stability and becomes hyperbolic with one multiplier smaller than -1 . The system changes orientation on the one-dimensional unstable manifold $W^u(K)$ of the hyperbolic point K . The manifold $W^u(K)$ ends in the 2-periodic stable orbit A . For $h > 4.1$, the 2-periodic orbit A loses stability and a period doubling bifurcation occurs. Most likely, the described period doubling bifurcation is repeated for sufficiently large h .

4. Investigation of the General Case

Consider the discrete system (2) and assume that the parameter m takes positive values. The fixed points of the system are given by the equalities $1 = xy^{m/s}z$ and $y = z$. Thus, the curve of equilibrium states has the form

$$K = \{(x, y, z) : xy^{m/s+1} = 1, y = z\}.$$

It is easy to see that the dynamical system with arbitrary m has a foliation with invariant layers as in the case $m = 0$. This foliation is defined as level surfaces of the function

$$U = \frac{x^b}{y^a}.$$

Each surface

$$W = \{(x, y, z) : x = hy^{a/b}\}, \quad h = \text{const},$$

is invariant for system (2). The system on the surface W is defined in the form

$$\begin{aligned} y_{n+1} &= y_n \exp(b(1 - hy_n^{a/b+m/s}z_n)), \\ z_{n+1} &= z_n \exp(c(y_n - z_n)), \end{aligned} \tag{9}$$

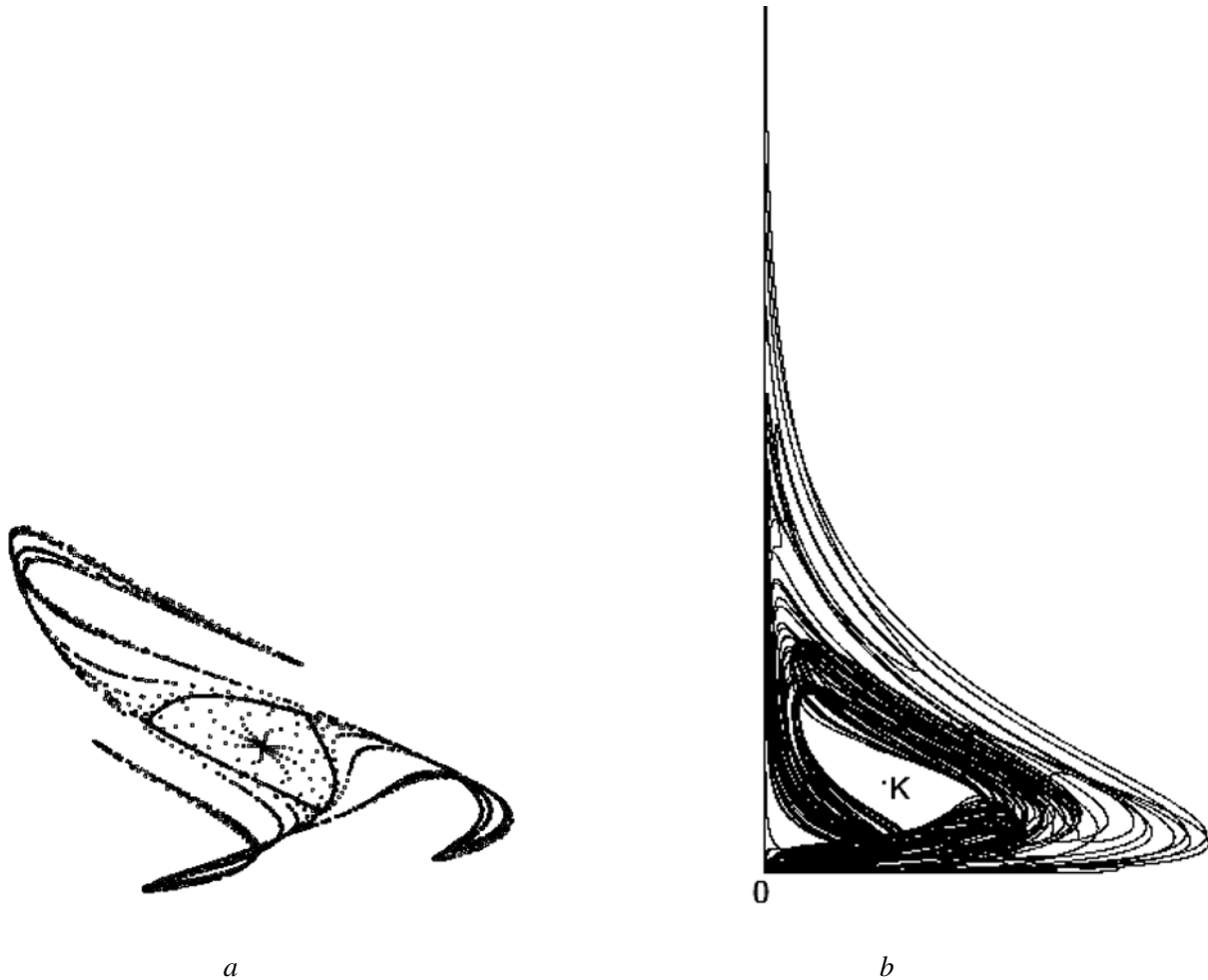


Fig. 5. Orbit of the point $(1.2; 1.2)$ in the invariant layer $W(h = 0.515970)$. Chaos in the layer $W(h = 0.3)$ on a 1:10 scale.

where the number $h > 0$ defines an invariant layer. Thus, system (2) has a foliation as in the case $m = 0$ but the dynamics in layers differs from the dynamics for $m = 0$.

We now study the distinctions observed for $m > 0$. For definiteness, we consider system (2) with $a = 2.45$, $b = 0.6$, $c = 0.9$, $m = 0.25$, and $s = 0.5$. The dynamics of the system near the curve of equilibrium states K is shown in Fig. 6. Each point of the curve K is associated with an invariant layer

$$W(h) = \left\{ (x, y, z) : \frac{x^b}{y^a} = h \right\}.$$

These layers are transverse to the curve K . Note that the layers are not shown in Fig. 6, and we present only periodic orbits stable in the layers. The equilibrium state $O(1, 1, 1)$ is hyperbolic. The unstable manifold $W^u(O)$ ending in a 2-periodic stable orbit is shown in Fig. 6a. In $W^u(O)$, the system changes orientation for each iteration. In layers below the layer of the point O , there appear period doubling bifurcations, which is well illustrated in Fig. 6a. In Fig. 6b, the dynamics in layer above the layer passing through the equilibrium state $O(1, 1, 1)$ is shown. As y and z increase, a bifurcation of equilibrium states occurs (as in the case $m = 0$). First, the equilibrium state becomes stable in the layer. This occurs in $W(h = 0.3083)$ where the equilibrium state has the coordinates $(1.234603; 1.234603)$.

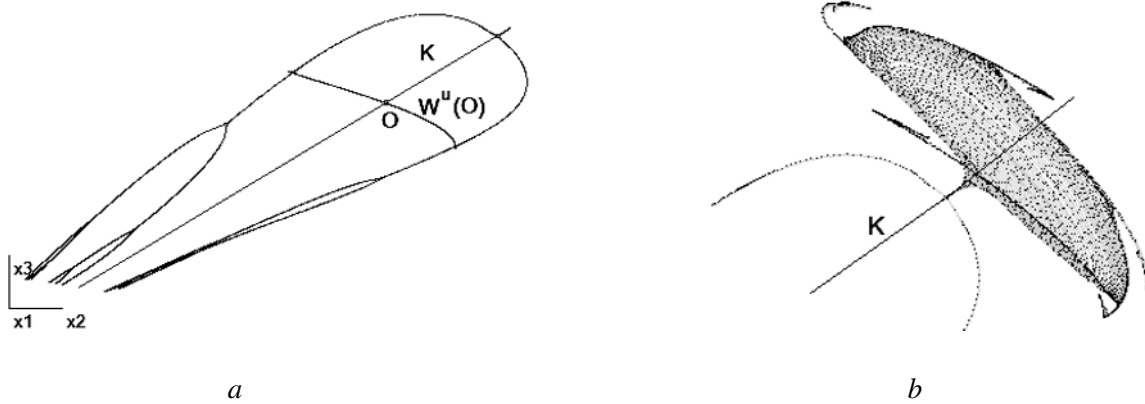


Fig. 6. Dynamics near the curve of equilibrium states K .

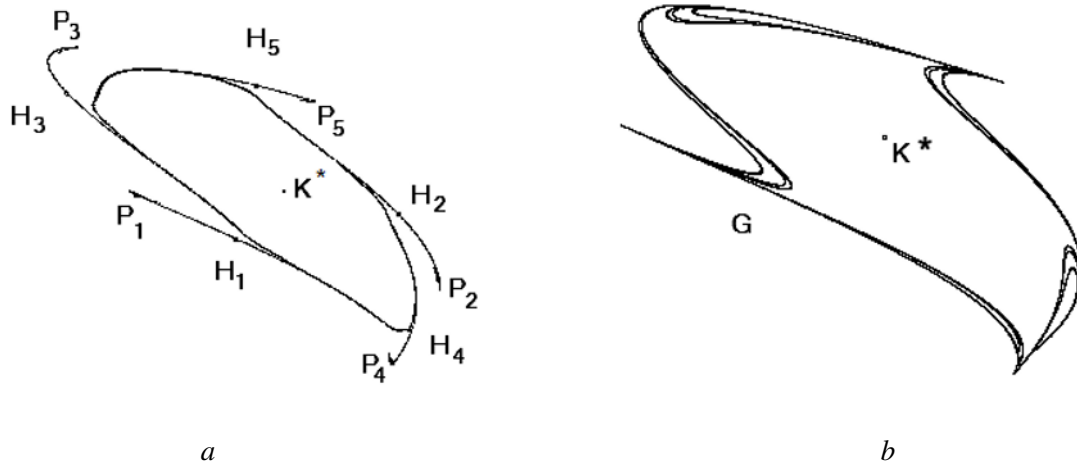


Fig. 7. Dynamics near the invariant ellipsoid in the layer $W(h = 0.18)$. The invariant set G in the layer $W(h = 0.155)$.

Then there appear complex multipliers in $W(h = 0.28)$. Finally, the equilibrium state loses stability in the layer $W(h = 0.2311)$ and we observe a Neimark–Sacker bifurcation. The dynamics near an invariant stable ellipse in the layer $W(h = 0.18)$ is shown in Fig. 7a. Here, K^* (1.359516; 1.359516) is the unstable equilibrium state, near the ellipse, the 5-periodic hyperbolic orbit H lies, and H_1 has the coordinates (1.289692; 1.235095). One end of the unstable manifold $W^u(H)$ tends to the stable ellipse, whereas the other end approaches the 5-periodic orbit P ; P_1 has the coordinates (1.156926; 1.338456) (see Fig. 7a). Moreover, the orbit starting near the point K^* ends on the ellipse.

A subsequent decrease in h leads to the bifurcation of the invariant ellipse into the set G depicted in Fig. 7b for the layer $W(h = 0.155)$. The invariant set G is limiting for any orbit starting near the point K . Estimating the entropy as the exponent of increase in the length of curve in the iteration, we reveal the chaotic behavior of the invariant set G . For $h = 0.15075$, the limiting set G of any orbit starting near the point K is large and the dynamics of the system becomes similar to the dynamics depicted in Fig. 5b.

5. Equivalence of Systems with $m \geq 0$ and $m = 0$

Comparing the dynamics of systems in the general case $m \geq 0$ and in the special case $m = 0$, we reveal certain similarity. Indeed, we reveal the topological equivalence of these systems but the parameter a should be changed for this purpose.

Proposition 2. *The mapping F of the form*

$$\begin{aligned} X &= xy^{m/s}, \\ Y &= y, \\ Z &= z \end{aligned} \tag{10}$$

maps the discrete system

$$\begin{aligned} x_{n+1} &= x_n \exp(a(1 - x_n y_n^{m/s} z_n)), \\ y_{n+1} &= y_n \exp(b(1 - x_n y_n^{m/s} z_n)), \\ z_{n+1} &= z_n \exp(c(y_n - z_n)) \end{aligned} \tag{11}$$

into the system

$$\begin{aligned} X_{n+1} &= X_n \exp(d(1 - X_n Z_n)), \\ Y_{n+1} &= Y_n \exp(b(1 - X_n Z_n)), \\ Z_{n+1} &= Z_n \exp(c(Y_n - Z_n)), \end{aligned} \tag{12}$$

where

$$d = a + \frac{bm}{s}.$$

Proof. It is necessary to prove the commutativity of the diagram

$$\begin{array}{ccc} (x_n, y_n, z_n) & \longrightarrow & (x_{n+1}, y_{n+1}, z_{n+1}) \\ F \downarrow & & F \downarrow \\ (X_n, Y_n, Z_n) & \longrightarrow & (X_{n+1}, Y_{n+1}, Z_{n+1}) \end{array}$$

or the equalities

$$X_{n+1}(F(x_n, y_n, z_n)) = F_x(x_{n+1}, y_{n+1}, z_{n+1}),$$

$$Y_{n+1}(F(x_n, y_n, z_n)) = F_y(x_{n+1}, y_{n+1}, z_{n+1}), \quad (13)$$

$$Z_{n+1}(F(x_n, y_n, z_n)) = F_z(x_{n+1}, y_{n+1}, z_{n+1}).$$

For the first equation, we find

$$\begin{aligned} X_{n+1} &= X_n \exp(d(1 - X_n Z_n)) = x_n y_n^{m/s} \exp((a + bm/s)(1 - x_n y_n^{m/s} z_n)) \\ &= x_n \exp(a(1 - x_n y_n^{m/s} z_n)) y_n^{m/s} \exp((bm/s)(1 - x_n y_n^{m/s} z_n)) = x_{n+1} y_{n+1}^{m/s}. \end{aligned}$$

For the second equation, we get

$$Y_{n+1} = Y_n \exp(b(1 - X_n Z_n)) = y_n \exp(b(1 - x_n y_n^{m/s} z_n)) = y_{n+1}.$$

The last equation is trivial because the map F is identical in y and z .

The proposition is proved.

Remark 1. The map F is a diffeomorphism in the domain $\{x > 0, y > 0, z > 0\}$, where systems (11) and (12) are given.

Thus, systems (11) and (12) are diffeomorphic. Moreover, for $a = 2.45$, $b = 0.6$, $c = 0.9$, $m = 0.25$, and $s = 0.5$, system (11) is equivalent to system (12), where $d = 2.75$, $b = 0.6$, and $c = 0.9$.

System (11) has a foliation with invariant layers of the form

$$W(h) = \left\{ (x, y, z) : h = \frac{x^b}{y^a} \right\},$$

and system (12) has a foliation with invariant layers of the form

$$W(H) = \left\{ (X, Y, Z) : H = \frac{X^b}{Y^d} \right\}.$$

It can be shown that the map F maps the layer $W(h)$ into the layer $W(H)$. Moreover, the systems on these layers coincide.

Taking into account that $x = P/P_e$, $y = (r_e/r)^s$, P is the price level, and r is the interest rate, for the introduced coordinate, we obtain

$$X = P/P_e ((r_e/r)^s)^{m/s} = \frac{P/r^m}{r_e^m/P_e},$$

i.e., from the economic point of view, the coordinate X is proportional to the price level and inversely proportional to the m th power of the interest rate.



Fig. 8. Orbit of the point $(1, 1, 1)$ of the perturbed system (3) with $\varepsilon = 0.01$. Orbit of the point $(1, 1, 1)$ of system (15) with $\varepsilon_1 = 0.01$ and $\varepsilon_{2,3} = 0$.

6. Uncontrollable Perturbations of the System

As a rule, economic systems suffer the action of uncontrollable and random perturbations. In this section, we study the dynamics of a perturbed system of the form (3), where ε is a small positive number,

$$q_i(n+1) = 1 - 2q_i^2(n), \quad (14)$$

and the initial values $q_i(0) \in [-1, 1]$ are randomly defined for each $i = 1, 2, 3$. Thus, we investigate a 6-dimensional system formed by the equations of system (3) and equations (14), $i = 1, 2, 3$. It is expected that the described perturbations do not preserve the invariant foliation. However, in each invariant layer of the unperturbed system, there exist attractors with certain domains of attraction which must be preserved under small perturbations. The results described above show that the attractors are formed from stable equilibrium states of the curve K . Indeed, the attractors appear as a result of the loss of stability of these equilibrium states. Moreover, the indicated attractors continuously vary from layer to layer. A fiber union of these attractors gives a set that does not disappear under perturbations. In Fig. 8a, this is well illustrated by the orbit of the point $(1, 1, 1)$ in the perturbed system (3) with the parameters $a = 2.45$, $b = 0.6$, $c = 0.9$, and $m = 0$ and the chaotic perturbation $\varepsilon = 0.01$. It is worth noting that, under the action of chaotic perturbations, the orbit moves up and down near the attractors of the unperturbed system. In this case, the perturbation is able not only to transfer an orbit into the chaotic domain (see the upper part in Fig. 8a) but also to return it from this domain.

From the economic point of view, there are both perturbations that can be made equal to zero or negligible and perturbations that cannot be significantly decreased. For example, the central bank may control the interest rate and does not allow chaotic perturbations. At the same time, it is impossible to remove chaotic perturbations of the price level. Thus, it makes sense to distinguish perturbations whose influence on the dynamics of the system is significant from perturbations that can be neglected. To this end, consider a system of equations of the form

$$x_{n+1} = x_n \exp(a(1 - x_n y_n^{m/s} z_n)) + \varepsilon_1 q_1(n),$$

$$y_{n+1} = y_n \exp(b(1 - x_n y_n^{m/s} z_n)) + \varepsilon_2 q_2(n), \quad (15)$$

$$z_{n+1} = z_n \exp(c(y_n - z_n)) + \varepsilon_3 q_3(n),$$

where ε_1 , ε_2 , and ε_3 are different and q_i satisfy system (14). Note that the perturbation of the last equation preserves invariant layers because each layer is a ruled surface parallel to the Z -axis. Thus, a perturbation of the form $\varepsilon_1 = 0$, $\varepsilon_2 = 0$, and $\varepsilon_3 \neq 0$ preserves the layers of the unperturbed system and perturbs the system in a layer. This means that the perturbation of the national income (production) weakly affects the dynamics near attractors in the layers.

A weak control over the price level leads to the perturbation of the first equation. The orbit of the point $(1, 1, 1)$ of system (15) with $\varepsilon_1 = 0.01$ and $\varepsilon_{2,3} = 0$ is shown in Fig. 8b. The numerical results demonstrate that the perturbations of the price level lead to the bifurcations of y and z . An increase in y leads to a decrease in the interest rate. It is worth noting that the strongest bifurcations into the chaotic state are observed for a weak control over the interest rate. The numerical experiments for system (15) with $\varepsilon_2 = 0.01$ and $\varepsilon_1 = \varepsilon_3 = 0$ show that this behavior of the system, in fact, does not differ from the behavior of a perturbed system of the general type (3) with $\varepsilon = 0.01$ (see Fig. 8a). Under small perturbations of the general type, the orbit begins to shift along the equilibrium states and then falls, first, into the unstable equilibrium state and then into the layer of chaos. The level of chaos can both increase and decrease under perturbations.

7. Conclusions

We study the behavior of the discrete macroeconomic model (2). It is shown that this system is C^1 -equivalent to system (2) with $m = 0$. It has a curve filled by equilibrium states and invariant level surfaces of the function

$$U = \frac{x^b}{y^a},$$

which form a foliation transverse to this curve. The equilibrium state can be regarded as the center of each invariant surface. Each layer contains an attractor and almost all orbits approach this attractor. This attractor can be either an equilibrium state or have a quite complicated (chaotic) structure. As the level surface varies, we observe a bifurcation of the dynamics of system from the stable equilibrium state to chaos. Note that chaos in the considered macroeconomic model is an inherent property of the system. Moreover, chaos does not always lead to the economic crisis. In the analyzed case, chaos means the impossibility of long-term predictions. The numerical results show that there are layers in which chaos is large, which leads to the disbalance of economic systems and the onset of crisis.

Small external perturbations can destroy the described topological structure of orbits in the system. The numerical experiments and economic practice show that not all perturbations identically affect the dynamics of the system. Thus, the perturbations of the national income (z) do not affect the invariant foliation, perturbations of the price level (x) lead to weak changes in the dynamics preserving attractors in the layers. At the same time, perturbations of the interest rate strongly affect the dynamics of the system. Under small perturbations, the orbit begins to move along equilibrium states and falls, first, into the unstable equilibrium state and then into the layer of chaos whose value may increase and become very large.

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