On the quantitative version of the Erdős - Turán conjecture about the additive representation functions

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Let A be a set of nonnegative integers. For a positive integer n let $R_A(n)$ denote the number of representations of n as the sum of two terms from A. One of the famous conjecture of Erdős and Turán asserts that if $R_A(n)$ is positive from a certain point on, then it cannot be bounded. There is a quantitative version of the Erdős - Turán conjecture formulated by Erdős and Fuchs. We improve a recent result of Haddad and Helou about the quantitative version of the Erdős - Turán conjecture.

Some explicit formulas of Bernoulli and Cauchy polynomials in terms of Stirling numbers

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The integral values of Bernoulli polynomials are expressed in terms of some extended Stirling numbers of the second kind. The integral values of Cauchy polynomials are expressed in terms of some extended Stirling numbers of the first kind. Several relations between the integral values of Bernoulli polynomials and those of Cauchy polynomials are obtained in terms of Stirling numbers of both kinds.

On sum sets of sets having small product set

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We improve a result of Solymosi on sum–products in $\mathbb{R}$, namely, we prove that $\max\{|A+A|, |AA|\} \gg |A|^{\frac{4}{3}+c}$, where $c > 0$ is an absolute constant. New lower bounds for sums of sets with small product set are found. Previous results are improved effectively for sets $A \subset \mathbb{R}$ with $|AA| \leq |A|^{4/3}$.

On the primitve of Hardy's function

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Let $Z(t)$ be Hardy's function, that is, $Z(t) = e^{i\vartheta(t)}\zeta\big(\frac{1}{2}+it\big)$, and let $F(T)$ be its integral:

$$F(T) = \int_{2\pi}^{T} Z(t)dt.$$

The estimate $F(T) \ll T^{7/8}$ was obtained by G.H. Hardy and J.E. Littlewood in proving the fact that $\zeta(s)$ has infinitely many zeros on the critical line. In 2004, A. Ivic proved that $F(T) \ll T^{1/4+\varepsilon}$ and conjectured that $F(T) = \Omega_{\pm}(T^{1/4})$. In 2007, the author proved that $|F(T)| <$

$18.2T^{1/4}$ for sufficiently large T and, moreover, that

$$F(T) \;=\; (-1)^N\sqrt{N}\mathfrak{K}(\alpha) \;+\; O\bigl(N^{1/3}\ln N\bigr),$$

where $T = 2\pi(N+\alpha)^2$, $N\in\mathbb{N}$, $0\leqslant\alpha<1$, that is,

$$N \;=\; \biggl[\sqrt{\frac{T}{2\pi}}\biggr], \quad \alpha = \biggl\{\sqrt{\frac{T}{2\pi}}\biggr\}.$$

Here $\mathfrak{K}(\alpha)$ denotes some function which satisfy to the equation $\mathfrak{K}(\alpha) = K(\alpha)$ for any fixed α, $0\leqslant\alpha<1$,

$$K(\alpha) \;=\; \begin{cases} 0, & \text{if } 0\leqslant\alpha<1/4,\ 3/4<\alpha<1,\\ 2\pi, & \text{if } 1/4<\alpha<3/4,\\ 4\pi/3, & \text{if } \alpha=1/4,\\ 2\pi/3, & \text{if } \alpha=3/4. \end{cases}$$

In particular, this proves the above conjecture of A. Ivic.

In 2009-2011, M. Jutila undertook a much more deep analysis of the behavior of the functions $F(T)$ and $\mathfrak{K}(\alpha)$. In particular, he obtained the following formula for $\mathfrak{K}(\alpha)$, which is uniform on α, $0\leqslant\alpha<1$:

$$\mathfrak{K}(\alpha) \;=\; K_0(\alpha) + \int_{-0.5}^{0.5} w(u)\beta(u)\bigl(K_0(\alpha+u)-K_0(\alpha)\bigr)du,$$

where $K_0(\alpha) = K(\alpha)$ for $\alpha\neq 1/4, 3/4$, $K_0(1/4) = K_0(3/4) = \pi$,

$$\beta(u) \;=\; \frac{1}{\pi}\int_0^{+\infty}\cos\bigl(Ax^3 - 2\pi xu\bigr)dx, \quad A = \frac{\pi}{12}\sqrt{\frac{2\pi}{T}},$$

is Airy function, $w(u)$ is a smooth weight function such that $w(u) = 1$ for $|u|\leqslant 1/4$, $w(u) = 0$ for $|u|\geqslant 1/2$ and $w^{(k)}(u)\ll_k 1$ for sufficiently many derivatives. M. Jutila also note that "...in a neighborhood of length about $T^{1/3}$ of any point $T = 2\pi(N+j/4)^2$ with N a natural number and $j=1$ or 3, the function $F(T)$ jumps an amount $\asymp T^{1/4}$ upwards and downwards and *the above* theorem moreover indicates how these jumps take place...".

In the talk, we present a new expression for the function $\mathfrak{K}(\alpha)$ which is free of smooth factor $w(u)$. The advantage of this expression is that it uncovers the features of the oscillations of $\mathfrak{K}(\alpha)$ near the points $1/4$ and $3/4$ and allow us to demonstrate the effects discovered by M. Jutila in a very precise form.

As a corollaries, we obtain a new unimprovable bound for $|F(T)|$, a new formula for the sum

$$\sum_{t_n\leqslant T} Z(t_n)$$

over Gram points t_n, and some other results.