Postclassical families of functions on descriptive and prescriptive spaces and their applications

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Descriptive spaces

A pair $(T, \mathcal{S})$, where T is a set and $\mathcal{S}$ is an non-empty family of subsets T (an *ensemble on* T) is called a *descriptive space*.
A multiplicative ensemble containing T and $\varnothing$ is called a *foundation*, a countably additive foundation is called a σ*-foundation*.
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$F(T)$ denotes the family of all real-valued functions on T.
$F_b(T)$ is its subfamily of all bounded functions.
$f^{-1}\left[X\right]$ denotes the preimage of the set $X \subset \mathbb{R}$ with respect to $f \in F(T)$.

The classical family of measurable functions

A function $f \in F(T)$ is called *measurable on a descriptive space* $(T, \mathcal{S})$ if $f^{-1}\left[\,]x, y[\,\right] \in \mathcal{S}$ for every open interval $]x, y[\subset \mathbb{R}$.
The family of all such functions is denoted by $M(T, \mathcal{S})$.
$M_b(T, \mathcal{S}) \equiv M(T, \mathcal{S}) \cap F_b(T)$.

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The Lebesgue – Borel theorem. *Let $\mathcal{S}$ be a σ-foundation on T. Then $M(T, \mathcal{S})$ and $M_b(T, \mathcal{S})$ contain $\mathbf{1} : T \to \{1\}$ and are closed with respect to addition, multiplication by real numbers, finite suprema and infima, multiplication, and uniform convergence. Moreover, $M(T, \mathcal{S})$ $[M_b(T, \mathcal{S})]$ is closed with respect to division on non-vanishing functions [by functions separated from zero by a constant, respectively].*

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Any family $A(T) \subset F(T)$ $[A(T) \subset F_b(T)]$ possessing all properties of $M(T, \mathcal{S})$ $[M_b(T, \mathcal{S})]$ from the Lebesgue – Borel theorem is called *normal* [*boundedly normal*, respectively].

The Lebesgue – Borel – Hausdorff theorem. *A family $A(T)$ is normal iff $A(T) = M(T, \mathcal{S})$ for some σ-foundation $\mathcal{S}$.*

The cover characterization of measurable functions

The number $\omega(f, A) \equiv \sup\{|f(t) - f(s)| \mid t, s \in A\}$ is called the *oscillation of a function* $f \in F(T)$ *on a set* $A \subset T$.
For a collection of subsets $\pi \equiv \big(A_i \subset T \mid i \in I\big)$ the number $\omega(f, \pi) \equiv \sup\big(\omega(f, A_i) \mid i \in I\big)$ is called the *oscillation of* f *on* π.
A collection $\pi \equiv \big(A_i \subset T \mid i \in I\big)$ such that $\bigcup\big(A_i \mid i \in I\big) = S$ is called a *cover of a set* S.

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Proposition 1. *Let* $\mathcal{S}$ *be a* σ*-foundation on* T *and* $f \in F(T)$*. Then* $f \in M(T, \mathcal{S})$ $[f \in M_b(T, \mathcal{S})]$ *iff for every* $\varepsilon > 0$ *there is a countable [finite] cover* $\pi \equiv \big(S_i \in \mathcal{S} \mid i \in I\big)$ *of* T *such that* $\omega(f, \pi) < \varepsilon$.

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The use of the "cover language" in Proposition 1 gives the basis for introducing new postclassical families of functions possessing good properties for any foundation $\mathcal{S}$, unlike the classical family of measurable functions has these properties only for a σ-foundation $\mathcal{S}$.

Postclassical families of functions

A function $f \in F(T)$ is called *distributable* [*uniform*] *on a descriptive space* $(T, \mathcal{S})$ if for every $\varepsilon > 0$ there is a countable [finite] cover $\pi \equiv \big(S_i \in \mathcal{S} \mid i \in I\big)$ of the set T such that $\omega(f, \pi) < \varepsilon$.
The family of all distributable [uniform] functions is denoted by $D(T, \mathcal{S})$ [$U(T, \mathcal{S})$]. Clearly, $U(T, \mathcal{S}) \subset F_b(T)$.

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Theorem 1. *Let $\mathcal{S}$ be a foundation on T. Then the family $D(T, \mathcal{S})$ is normal and the family $U(T, \mathcal{S})$ is boundedly normal,* i. e., *they possess all properties of the families $M(T, \mathcal{S})$ and $M_b(T, \mathcal{S})$ from the Lebesgue – Borel theorem, respectively.*

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Not all postclassical families of functions on a descriptive space are reducible to the classical ones.
Proposition 2. *There exists a foundation $\mathcal{K}$ on $T \equiv [0, 1[\subset \mathbb{R}$ such that for every σ-foundation $\mathcal{S}$ on T the inequality $U(T, \mathcal{K}) \neq M_b(T, \mathcal{S})$ holds.*

Prescriptive spaces

Descriptive spaces are not a “native habitat” for the postclassical families, in contrast to the classical family of measurable functions. For this reason, below we consider postclassical families of uniform and distributable functions on prescriptive spaces natural for them.

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A pair $(T, \mathfrak{C})$, where T is a set and $\mathfrak{C}$ is a family of covers $\pi \equiv \big(A_i \subset T \mid i \in I\big)$ of T (a *covering on* T) containing the *one-member cover* $\big(A_i \equiv T \mid i \in \{i\}\big)$, will be called a *prescriptive space*.

A covering $\mathfrak{C}$ is called *multiplicative* if for any collections $\pi \equiv \big(A_i \mid i \in I\big) \in \mathfrak{C}$ and $\rho \equiv \big(B_j \mid j \in J\big) \in \mathfrak{C}$ the collection $\pi \wedge \rho \equiv \big(C_k \mid k \in K\big)$, where $K \equiv I \times J$ and $C_k \equiv A_i \cap B_j$ for every $k \equiv (i, j) \in K$, belongs also to $\mathfrak{C}$.

Distributable functions on prescriptive spaces

A function $f \in F(T)$ is called *distributable on the prescriptive space* $(T, \mathfrak{C})$ if for every $\varepsilon > 0$ there is countable cover $\pi \in \mathfrak{C}$ of the set T such that $\omega(f, \pi) < \varepsilon$.
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If $\mathcal{S}$ is a foundation, then covering $\operatorname{Cov}_c \mathcal{S}$ consisting of all countable covers $\big(S_i \in \mathcal{S} \mid i \in I\big)$ of T is multiplicative.
The family $D(T, \operatorname{Cov}_c \mathcal{S})$ was denoted above by $D(T, \mathcal{S})$.

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Theorem 2. *Let $\mathfrak{C}$ be a multiplicative covering on T. Then $D(T, \mathfrak{C})$ possesses all the properties of $M(T, \mathcal{S})$ from the Lebesgue – Borel theorem except closedness with respect to multiplication and division by nonvanishing functions.*
Under some additional conditions on $\mathfrak{C}$ the family $D(T, \mathfrak{C})$ has these two properties as well.

Uniform functions on prescriptive spaces

A function $f \in F(T)$ is called *uniform on the prescriptive space* $(T, \mathfrak{C})$ if for every $\varepsilon > 0$ there is a finite cover $\pi \in \mathfrak{C}$ of the set T such that $\omega(f, \pi) < \varepsilon$.
The set of all such functions is denoted by $U(T, \mathfrak{C})$.
It is easy to see that $U(T, \mathfrak{C}) \subset F_b(T)$.

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Theorem 3. *Let $\mathfrak{C}$ be a multiplicative covering on T. Then $U(T, \mathfrak{C})$ is boundedly normal,* i. e., *it has all the properties of $M_b(T, \mathcal{S})$ from the Lebesgue – Borel theorem.*

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The following theorem shows that the families of uniform functions on prescriptive spaces are as natural as the families of measurable functions on descriptive spaces.

Theorem 4. *A family $A(T) \subset F_b(T)$ is boundedly normal iff $A(T) = U(T, \mathfrak{C})$ for some multiplicative covering $\mathfrak{C}$.*

Application I: Hausdorff–Sierpiński problem

Let $(T, \mathcal{G})$ be a Hausdorff topological space with the ensembles $\mathcal{G}$ and $\mathcal{F}$ of open and closed sets, respectively.
If $(T, \mathcal{G})$ is not a Tychonoff (completely regular) space, then the space $C(T, \mathcal{G}) = M(T, \mathcal{G})$ of continuous functions may contain only constant functions.

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The families $SC^u(T, \mathcal{G})$ and $SC^l(T, \mathcal{G})$ of upper and lower semicontinuous functions are non-trivial for any Hausdorff space. In particular, they contain functions $\chi_{\{t\}}$ and $\chi_{T\setminus\{t\}}$ for every $t \in T$, respectively.

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These families as their subfamilies $SC_b^u(T, \mathcal{G})$ and $SC_b^l(T, \mathcal{G})$ of bounded functions are uniformly closed (closed with respect to uniform convergence), but they are not linear spaces (they are only cones).
Their linear envelopes $H(T, \mathcal{G}) \equiv SC^l(T, \mathcal{G}) + SC^u(T, \mathcal{G})$ and $H^b(T, \mathcal{G}) \equiv SC_b^l(T, \mathcal{G}) + SC_b^u(T, \mathcal{G})$ are linear spaces but they are not uniformly closed.

Application I: Hausdorff – Sierpiński problem

Due to the Lebesgue – Borel – Hausdorff theorem we know that the smallest normal family containing $H(T, \mathcal{G})$ (*its normal envelope*) is the classical family of measurable functions $M(T, \mathcal{S})$ for some σ-foundation $\mathcal{S}$. Clearly, $\mathcal{S} \supset \mathcal{G}$.

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Theorem 5. *The boundedly normal envelope of* $H_b(T, \mathcal{G})$ *is the postclassical family* $U(T, \mathcal{K})$ *of uniform functions with the foundation* $\mathcal{K} \equiv \{G \cap F \mid G \in \mathcal{G} \,\&\, F \in \mathcal{F}\}$.

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Note that $\mathcal{S} = \mathcal{K}_\sigma$.

Application II: the Riesz – Radon – Fréchet problem

The postclassical family $U(T, \mathcal{K})$ plays also the key role in solving the problem of characterization of Radon integrals as linear functionals for an arbitrary Hausdorff space $(T, \mathcal{G})$. This problem originates in the well-known Riesz representation theorem on characterization of Riemann – Stiltjes integrals as bounded linear functionals.

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For various classes of spaces $(T, \mathcal{G})$ this problem was solved by J. Radon (1913), S. Saks (1937), S. Kakutani (1941), P. Halmos (1950), E. Hewitt (1952), N. Bourbaki (1969) by means of the classical family $C(T, \mathcal{G})$.

This classical family is not appropriate in the general case because it may contain only constant functions, and, therefore, does not separate Radon measures ($\mu \neq \nu$ but $\int f\, d\mu = \int f\, d\nu$ on $C(T, \mathcal{G})$).

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Using uniform functions we obtain, e. g., the following result.
Theorem 6. *The mapping* $\mu \mapsto \int \cdot\, d\mu | U(T, \mathcal{K})$ *is a bijection (moreover, an isomorphism) between the lattice linear spaces of bounded Radon measures and of* σ*-exact linear functionals on* $U(T, \mathcal{K})$.

Application III: new characterization of Riemann integrable functions

Let $(T, \mathcal{G})$ be a Tychonoff space and μ be a positive bounded Radon measure on it. The Riemann integral for functions $f : T \to \mathbb{R}$ was defined by N. Bourbaki.

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Let $\mathcal{N}_\mu \equiv \{N \subset T \mid \exists\, F \in \mathcal{F}_\mu\ (N \subset F)\}$, where $\mathcal{F}_\mu$ is the ensemble of all closed null sets, and $\mathcal{ZP}_\mu \equiv \{G \cup N \mid G \in \mathcal{G} \,\&\, N \in \mathcal{N}_\mu\}$.

Theorem 6. *Let $f \in F(T)$. Then f is Riemann integrable on $(T, \mathcal{G})$ iff $f \in U(T, \mathcal{ZP}_\mu)$.*

This assertion gives new internal description of Riemann integrable functions different from the Lebesgue description.
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Thank you!