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Exact solutions of the Burgers–Huxley equation via dynamics

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ABSTRACT

According to the results of B. Kruglikov, O. Lychagina, and V. Lychagin, evolutionary partial differential equations determine flows (finite-dimensional dynamics) on solutions' spaces of ordinary differential equations.

In the present paper, we construct such dynamics for the classical Burgers–Huxley equation and then we use them to construct new exact solutions.

1. Introduction

The Burgers–Huxley equation

$$u_t + uu_x = u_{xx} + f(u) \tag{1}$$

is known in various fields of applied mathematics. For example, it describes transport processes in systems when diffusion and convection are equally important [19] and nonlinear reaction–diffusion processes too [18].

For example, the reaction–diffusion equation of with a nonlinear convection flow $H(u)$ in the positive direction of the x-axis has the form [13]

$$u_t + \frac{\partial H(u)}{\partial x} = u(1-u) + u_{xx}.$$

In the case when $H(u) = \dfrac{u^2}{2}$, this equation takes form (1).

The role of the convection mechanism in the predator–prey systems is considered in [6,20].

It is usually assumed that the function f is a polynomial of the second or third degree. If $f(u) = 0$ then Eq. (1) is well known as the Burgers equation. In the absence of the convection term uu_x, (1) is known as the Kolmogorov–Petrovsky–Piskunov equation [7].

There are a large number of works devoted to numerical and analytical methods for the Burgers–Huxley equations (see, for example, [5,14,17,19]). In particular, a group analysis of the equations was carried out and their reductions were constructed [3].

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In the present article, the theory of finite-dimensional dynamics is used in order to construct exact solutions of Eq. (1) with a cubic term $f(u)$. This allowed us to construct new exact solutions even in the cases when the equation does not have the necessary algebra of symmetries.

The theory of finite-dimensional dynamics is a natural development of the theory of dynamical systems. Its methods make it possible to construct finite-dimensional submanifolds in an infinite-dimensional space among solutions of the equations. Elements of these submanifolds are numerated by the solutions of ordinary differential equations.

The basic ideas and methods of this theory were formulated in [9,12]. In the same papers, finite dynamics were constructed for the Kolmogorov–Petrovsky – Piskunov and for the Korteweg de Vries and equations.

Dynamics of third order were found for equations of the Rapoport–Leas type arising in the theory of two-phase filtration. These dynamics were used for constructing attractors [1,2].

The structure of this article is as follows.

In the first section we give basic definitions and describe methods of the theory. Details can be found in [4,9,10,12].

The second section presents the dynamics of the first order. They are used to construct exact solutions of the Burgers–Huxley. First-order dynamics we construct in the form of Riccati equations. This provides wide-coverage of cubic nonlinearities.

In the third section, we focus on second-order dynamics. Such dynamics we construct in the form of Liénard equations [11]. Integration of the obtained Liénard equations is ensured by the presence of two shuffling symmetries, which allows us to use the Lie–Bianchi theorem [4,10].

2. Evolutionary PDEs as dynamics on ODEs' solutions

Consider the ordinary differential equation of kth order

$$y^{(k+1)} = h\left(x, y, y', y'', \dots, y^{(k)}\right). \tag{2}$$

This equation defines a one-dimensional distribution P on the space $J^k(\mathbb{R})$ of k-jets. Prolongations of solutions' graphs into the space $J^k(\mathbb{R})$ are integral curves of this distribution. The distribution P is defined by the vector field

$$\mathcal{D} = \frac{\partial}{\partial x} + y_1 \frac{\partial}{\partial y_0} + \dots + y_k \frac{\partial}{\partial y_{k-1}} + h\frac{\partial}{\partial y_k}.$$

Here $x, y_0, y_1, \dots, y_k$ are canonical coordinates on $J^k(\mathbb{R})$ [8].

A vector field X on $J^k(\mathbb{R})$ is called an *infinitesimal symmetry* of Eq. (2) if translations along X save P.

All infinitesimal symmetries form the Lie algebra with respect to the Lie bracket which we denote by $\mathsf{Symm}\,\mathsf{P}$.

An infinitesimal symmetry is called *characteristic* if translations along it save each integral curve of the distribution P.

Characteristic symmetries form an ideal in $\mathsf{Symm}\,\mathsf{P}$ which we denote by $\mathsf{Char}\,\mathsf{P}$. The quotient Lie algebra

$$\mathsf{Shuff}\,\mathsf{P} := \mathsf{Symm}\,\mathsf{P}/\mathsf{Char}\,\mathsf{P}$$

is called the Lie algebra of *shuffling* symmetries.

Elements of this Lie algebra can be presented by vector fields which are transversal to the distribution P: each shuffling symmetry can be identified with a vector field of the form

$$S_\phi = \phi\frac{\partial}{\partial y_0} + \mathcal{D}(\phi)\frac{\partial}{\partial y_1} + \mathcal{D}^2(\phi)\frac{\partial}{\partial y_2} + \dots + \mathcal{D}^k(\phi)\frac{\partial}{\partial y_k},$$

where ϕ is a function on $J^k(\mathbb{R})$ that is called a *generating function* of the corresponding shuffling symmetry.

Trajectories of the vector field S_ϕ can be found as solutions of the following system of ordinary differential equations:

$$\begin{cases} \dfrac{dy_0}{dt} = \phi, \\ \dfrac{dy_1}{dt} = \mathcal{D}(\phi), \\ \dots\dots\dots\dots \\ \dfrac{dy_k}{dt} = \mathcal{D}^k(\phi). \end{cases}$$

Thus, if $y = y(x)$ is a solution of Eq. (2), then the function

$$u(t, x) = \left(\Phi_t^{-1}\right)^* (y(x))$$

is a solution of the following evolutionary partial differential equation:

$$\frac{\partial u}{\partial t} = \phi\left(x, u, \frac{\partial u}{\partial x}, \dots, \frac{\partial^k u}{\partial x^k}\right) \tag{3}$$

with the initial data $u(0, x) = y(x)$. Here Φ_t is the translation along the vector field S_ϕ from $t = 0$ to t.

On the other hand, suppose that evolutionary equation (3) is given and we know ordinary differential equation (2) for which the function $\phi = \phi(x, y_0, \dots, y_k)$ is a generating function of a shuffling symmetry. Shifting solutions of (2) along the vector field S_ϕ from $t = 0$ to t, we obtain the solutions of evolutionary equation (3).

If ϕ is a generating function of some symmetry then Eq. (2) is called a (finite-dimensional) *dynamics* of Eq. (3). The number $k + 1$ is called an *order* of the dynamics.

Thus, an evolutionary equation determines a flow on the solutions' space of an ordinary differential equation.

The following theorem (see [2]) provides a method for calculating finite-dimensional dynamics of evolutionary equations.

Theorem 1. *The ordinary differential equation*

$$F = y_{k+1} - h(x, y_0, y_1, \dots, y_k) = 0$$

is a dynamics of evolutionary equation (3) if and only if

$$[\phi, F] = 0 \bmod \mathbf{DF}, \tag{4}$$

where $\mathbf{DF} = \langle F, D(F), D^2(F), \dots \rangle$ *is the generated by the function* F *differential ideal,*

$$D = \frac{\partial}{\partial x} + y_1 \frac{\partial}{\partial y_0} + y_2 \frac{\partial}{\partial y_1} + \cdots$$

is the operator of total derivative, and

$$[\phi, F] = \sum_{i=0}^{k} \left(\frac{\partial \phi}{\partial y_i} D^i(F) - \frac{\partial F}{\partial y_i} D^i(\phi) \right)$$

is a prolongation of the classical Poisson–Lie bracket into the jet space (see, for example, [10]).

Note that the Poisson–Lie bracket is skew-symmetric, $\mathbb{R}$-bilinear, and satisfies the Jacobi identity.

3. First order dynamics of the Burgers–Huxley equation

Construct first order dynamics of Eq. (1) in the following form:

$$y' = h(y), \tag{5}$$

where h is some smooth function. Then $F = y_1 - h(y_0)$ and the Poisson–Lie bracket has the form

$$[\phi, F] = y_1 h(y_0) - f'(y_0)h(y_0) - h''(y_0)y_1^2 + h'(y_0)f(y_0).$$

Then Eq. (4) takes the form

$$h^2 h'' + f'(y_0)h - f(y_0)h' - h^2 = 0. \tag{6}$$

This equation has the trivial solution $h = 0$ which corresponds to x-independent solutions of Eq. (1). We are not considering this case. Then Eq. (6) can be written in form of the Abel differential equation of second kind

$$hh' + (\alpha - y_0)h + f(y_0) = 0, \tag{7}$$

where α is a constant.

3.1. The function f is quadratic

Let the function f be quadratic:

$$f(u) = f_2 u^2 + f_1 u + f_0, \qquad f_2 \neq 0.$$

We will search for h as a linear function:

$$h(y) = Ay + B$$

with $A \neq 0$. From (7) it follows that $A = f_2$ and B is a root of the following quadratic equation:

$$B^2 - f_1 B + f_0 f_2 = 0.$$

It has real roots if $f_1^2 - 4f_0 f_2 \geq 0$. In this case

$$B = \frac{f_1 \pm \sqrt{f_1^2 - 4f_2 f_0}}{2}. \tag{8}$$

The general solution of Eq. (5) is

$$y(x) = -\frac{B}{A} + C_1 \exp(Bx), \tag{9}$$

where C_1 is arbitrary constant. Restrict the vector field S_ϕ to Eq. (5)[1]:

$$\overline{S_\phi} = \left((f_1 + f_2^2 - B)y_0 + f_0 + f_2 B\right) \frac{\partial}{\partial y_0}.$$

This vector field generates transformations

$$\Phi_t : \begin{cases} x \longmapsto x, \\ y_0 \longmapsto \dfrac{(ay_0 - f_0 - f_2 B)\exp(-at) + f_0 + f_2 B}{a}, \end{cases}$$

where $a = B - f_2^2 - f_1$.

Applying this transformation to function (9), we obtain a solution of Eq. (1):

$$u(t, x) = \frac{1}{af_2}((f_2 C_1 a \exp(f_2 x) - f_2 f_0 - B(B - f_1))\exp(-at) + f_2(f_0 + f_2 B)).$$

Here B is defined by formula (8).

3.2. The function f is a third order polynomial

Consider the case when the function f is a polynomial of degree 3:

$$f(u) = \sum_{i=0}^{3} f_i u^i, \quad f_0, \ldots, f_3 \in \mathbb{R}, \quad f_3 \neq 0.$$

Usually this kind of function is implied when talking about the Burgers–Huxley equation. In this case, we can search for first-order dynamics (5) with the quadratic function

$$h(y) = Ay^2 + By + C, \tag{10}$$

where coefficients A, B, C can be found from Eqs. (6) or (7). In this case Eq. (5) is the Riccati equation

$$y' = Ay^2 + By + C. \tag{11}$$

If $A = 0$ and $B \neq 0$ then from equality (6) it follows that $f_3 = 0$ and we come back to the case just considered. Therefore, we assume that $A \neq 0$. Denote $D = B^2 - 4AC$. The type of solution of Eq. (11) depends on the sign of the number D.

$$\text{If } D < 0 \text{ then} \quad y(x) = -\frac{B}{2A} - \frac{\sqrt{-D}}{2A} \tan\left(\frac{\sqrt{-D}(x + C_1)}{2}\right), \tag{12}$$

$$\text{If } D = 0 \text{ then} \quad y(x) = -\frac{B}{2A} - \frac{1}{A(x + C_1)}, \tag{13}$$

$$\text{If } D > 0 \text{ then} \quad y(x) = -\frac{B}{2A} - \frac{\sqrt{D}}{2A} \tanh\left(\frac{\sqrt{D}(x + C_1)}{2}\right), \tag{14}$$

where C_1 is an arbitrary constant.

Remark. Note that this kind of function h does not coverage of all polynomials of the third degree. As we will see, sometimes coefficients of the polynomial must be limited by inequalities.

Below we consider several types of polynomials of the third degree that have nontrivial first order dynamics. For each of them, we obtain exact solutions of the Burgers–Huxley equation.

Type 1. The polynomial f has a zero root. In this case we have the classical Burgers–Huxley equation with

$$\boxed{f(u) = au^3 + bu^2 + cu} \tag{15}$$

with constant a, b, c.

[1] Here and below, the upper bar means the restriction on the corresponding dynamic.

Eq. (6) is equivalent to the following system of equations with respect to A, B, C:

$$\begin{cases} C((1-2A)C-c)=0, \\ C(2AB-B+b)=0, \\ B(a-A+2A^2)=0, \\ B^2-4A^2C+(c-2B^2+2C)A-3aC-Bb=0, \\ A(a+A-2A^2)=0. \end{cases} \tag{16}$$

The last equation has a non-zero real solution only if

$$a \le \frac{1}{8}. \tag{17}$$

Assume that this inequality is satisfied. If we put

$$A=\frac{1-\sqrt{1-8a}}{4}, \quad B=\frac{2b}{1+\sqrt{1-8a}}, \quad C=\frac{2c}{1+\sqrt{1-8a}}$$

then equalities (16) hold for any a, b, c.

Three cases can be realized:

I. If $b^2-4ac<0$ then the function

$$y(x)=-\frac{b}{2a}+\frac{\sqrt{4ac-b^2}}{2a}\tan\left(\frac{\sqrt{4ac-b^2}(x+C_1)}{1+\sqrt{1-8a}}\right) \tag{18}$$

is a solution of Eq. (12).

II. If $b^2-4ac=0$ then the function

$$y(x)=-\frac{b}{2a}-\frac{1+\sqrt{1-8a}}{2a(x+C_1)} \tag{19}$$

is a solution of Eq. (13).

III. If $b^2-4ac>0$ then the function

$$y(x)=-\frac{b}{2a}-\frac{\sqrt{b^2-4ac}}{2a}\tanh\left(\frac{\sqrt{b^2-4ac}(x+C_1)}{1+\sqrt{1-8a}}\right) \tag{20}$$

is a solution of Eq. (14).

Consider these cases sequentially.

I. The restriction of the vector field S_ϕ to Eq. (11) is

$$\overline{S_\phi}=\frac{4b(ay_0^2+by_0+c)}{1+\sqrt{1-8a}}\frac{\partial}{\partial y_0}.$$

It generates the following transformation of y_0:

$$\Phi_t : y_0 \mapsto -\frac{b}{2a}+\frac{\sqrt{4ac-b^2}}{2a}\tan\left(\arctan\frac{2ay_0+b}{\sqrt{4ac-b^2}}-\frac{tb\sqrt{4ac-b^2}}{4a-1-\sqrt{1-8a}}\right).$$

Applying this transformation to (18), we obtain the following solution of Eq. (1):

$$u(t,x)=-\frac{b}{2a}-\frac{\sqrt{4ac-b^2}}{2a}\tan\frac{\sqrt{4ac-b^2}\left((1+\sqrt{1-8a})(x+bt)-4ax\right)}{2(6a-1+(2a-1)\sqrt{1-8a})}.$$

Here and below we put $C_1=0$.

II. The vector field

$$\overline{S_\phi}=\frac{b(4a^2y_0^2+4aby_0+b^2)}{a(1+\sqrt{1-8a})^2}\frac{\partial}{\partial y_0}$$

generates the following transformation:

$$\Phi_t : (x, y_0) \longmapsto \left(x, \quad \frac{2a(b^2t+1+\sqrt{1-8a}-4a)y_0+b^3t}{2a(1-2abty_0-4a-b^2t+\sqrt{1-8a})}\right).$$

Applying this transformation to (19), we obtain the following solution of the equation:

$$u(t,x)=\frac{(b^2\sqrt{1-8a}+b^2)t+(bx+2-4a)\sqrt{1-8a}-12a+(1-4a)bx+2}{2a(bt+x)\sqrt{1-8a}+2a(4ax-bt-x)}.$$

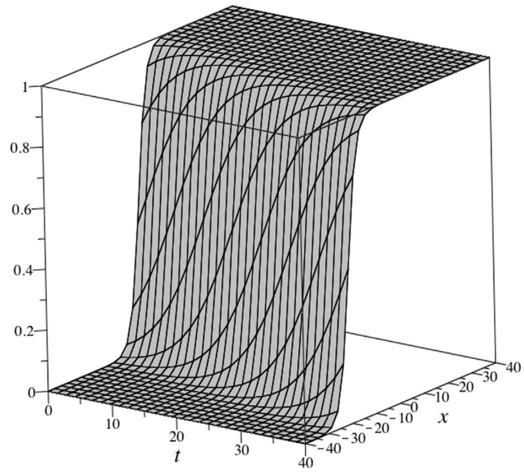

Fig. 1. The graph of the kink-solution, $a = -2, b = 3/2, c = 0$.

III. In this case, the vector field $\overline{S_\phi}$ is the same as in the case I, but the transformations are differ:

$$\Phi_t : y_0 \longmapsto -\frac{b}{2a} - \frac{\sqrt{D}}{2a}\tanh\left(\frac{(4a-1-\sqrt{1-8a})\ln\left(\dfrac{-b+\sqrt{D}-2ay_0}{b+\sqrt{D}+2ay_0}\right) - 2tb\sqrt{D}}{2-2\sqrt{1-8a}+8a}\right).$$

Applying this transformation to (20), we obtain the following solution of Eq. (1):

$$u(t,x) = \frac{-\left(b+\sqrt{D}\right)\exp(v(t,x)) - b + \sqrt{D}}{2a\exp(v(t,x))+1}, \tag{21}$$

where

$$v(t,x) = \frac{-2(bt+x)\sqrt{1-8a} + 8ax - 2(bt+x)\sqrt{D} + 8a - \sqrt{1-8a} - 1}{2(2a-1)\sqrt{1-8a} - 2 + 12a}.$$

Example 1. Solution (21) takes more simple form if we put $a = -2$, $b = 3/2$, $c = 0$. In this case

$$u(t,x) = \frac{\exp\left(\dfrac{t}{4}+\dfrac{x}{2}+\dfrac{3}{8}\right)}{\exp\left(\dfrac{t}{4}+\dfrac{x}{2}+\dfrac{3}{8}\right)+1} \tag{22}$$

The graph of solution (22) has form of a kink (see Fig. 1). Note that for solution (22)

$$\lim_{t\to+\infty} u(t,x) = \frac{3}{4}$$

and this solution can be considered as a model of reaching the stationary mode.

Remark. The classical Burgers–Huxley equation

$$u_t + uu_x = u_{xx} + au^3 + bu^2 + cu$$

with arbitrary coefficients a, b, c has only translational symmetries in variables t and x (see [3]). The ordinary differential equation corresponding to these symmetries turns out to be rather complicated and cannot be integrated in quadratures.

Type 2. One real non-zero and two complex roots. In this case

$$\boxed{f(u) = (au+b)\,(u^2+c^2)} \tag{23}$$

for some constant a, b, c and $a, b \neq 0$.

If $a < 1/8$ then the number a can be represented as

$$a = -\frac{A}{(A-2)^2}$$

for some real number A and we can put

$$f(u) = \left(-\frac{Au}{(A-2)^2} + b\right)(u^2 + c^2).$$

Solving Eq. (7) we get:

$$h(y) = \frac{A(y^2 + c^2)}{2(A-2)}$$

and the general solution of Eq. (11) is

$$y(x) = c\tan\frac{Ac(x + C_1)}{2A - 4} \tag{24}$$

where C_1 is an arbitrary constant. The restriction of the function ϕ to Eq. (5) is $\overline{\phi} = b(y_0 + c^2)$ and the translation transformation along the vector field $\overline{S_\phi}$ is

$$\Phi_t : (x, y_0) \longmapsto \left(x, \quad c\tan\left(bct + \arctan\frac{y_0}{c}\right)\right).$$

Applying this transformation to (24), we obtain the following solution of Eq. (1):

$$u(t,x) = c\tan\left(\frac{Acx}{2A-4} + bct + q\right),$$

where q is an arbitrary constant.

Type 3. Three real different roots and the coefficient at the senior degree is -1. In this case we can put

$$\boxed{f(u) = -(u-a)(u^2 + bu + c)} \tag{25}$$

with $b^2 - 4c > 0$. Then

$$h(y) = -\frac{y^2 + by + c}{2}$$

and the general solution of Eq. (11) has a form of a kink:

$$y(x) = -\frac{b}{2} + \frac{\sqrt{b^2 - 4c}}{2}\tanh\left(\frac{\sqrt{b^2-4c}(x + C_1)}{4}\right),$$

where C_1 is arbitrary constant. The restriction of the function ϕ to Eq. (5) is

$$\overline{\phi} = \frac{1}{2}\left(\frac{b}{4} + a\right)(y_0^2 + by_0 + c).$$

and the translation transformation along the vector field $\overline{S_\phi}$ is

$$\Phi_t : y_0 \longmapsto -\frac{b}{2} - \frac{\sqrt{b^2-4c}}{2}\tanh\left(\frac{1}{2}\ln\frac{-b + \sqrt{b^2-4c} - 2y_0}{b + \sqrt{b^2-4c} + 2y_0} + \frac{t\sqrt{b^2-4c}(4a+b)}{8}\right).$$

We obtain a solution of Eq. (1):

$$u(t,x) = -\frac{\left(-1 + \tanh\frac{\omega x}{4}\right)(\omega + b)\exp\left(\frac{(4a+b)\omega t}{4}\right) + \left(1 + \tanh\frac{\omega x}{4}\right)(\omega - b)}{2\left(-1 + \tanh\frac{\omega x}{4}\right)\exp\left(\frac{(4a+b)\omega t}{4}\right) - 2\left(1 + \tanh\frac{\omega x}{4}\right)},$$

where $\omega = \sqrt{b^2 - 4c}$.

Type 4. One real root of multiplicity 3. In this case

$$\boxed{f(u) = a(u-b)^3.} \tag{26}$$

Suppose that $a < 1/8$ and $a \neq 0$. Then the function (10) with

$$A = \frac{1 - \sqrt{1 - 8a}}{4}, \tag{27}$$

$$B = -\frac{2ab(1+\sqrt{1-8a})}{1-4a+\sqrt{1-8a}}, \tag{28}$$
$$C = \frac{b^2(1-\sqrt{1-8a})(-1+4a-\sqrt{1-8a})}{16a-4(1+\sqrt{1-8a})} \tag{29}$$

is a solution of Eq. (7) if

$$\alpha = -\frac{4ab(1+\sqrt{1-8a})}{(1-4a+\sqrt{1-8a})(-1+\sqrt{1-8a})}.$$

The general solution of Eq. (5) is

$$y(x) = \frac{(abx-1)\sqrt{1-8a}-1+a(bx+4)}{a(1+\sqrt{1-8a})x}, \tag{30}$$

where C_1 is arbitrary constant. The translation transformation along the vector field $\overline{S_\phi}$ is

$$\Phi_t : (x, y_0) \mapsto \left(x, \frac{bPt+Qy_0}{Pt+Q}\right),$$

where

$$P = 2ab\left(\left(a-\frac{1}{2}\right)\sqrt{1-8a}+3a-\frac{1}{2}\right)(b-y_0), \tag{31}$$
$$Q = 1-8a+\sqrt{1-8a}+8a^2-4a\sqrt{1-8a}. \tag{32}$$

Corresponding solution of Eq. (1) is

$$u(t,x) = b + \frac{\beta}{32a(x-bt)},$$

where

$$\beta = \frac{(1+16a^4-80a^3+60a^2-14a)\sqrt{1-8a}+1+144a^4-240a^3+108a^2-18a}{\left(-\frac{3a}{8}+\frac{5a^2}{4}-a^3+\frac{1}{32}\right)\sqrt{1-8a}-4a^3+\frac{1}{32}+\frac{5a^2}{2}-\frac{a}{2}+a^4}.$$

Rational function Consider one case when the function f is fractional-rational:

$$\boxed{f(u) = \frac{6u^3+au^2+4}{9u^3}}$$

with real a. Eq. (5) takes the form

$$y' = \frac{2}{3y}.$$

Its general solution is

$$y(x) = \pm\frac{1}{3}\sqrt{12x+C},$$

where C is an arbitrary constant. Shifting this solution along the trajectories of the vector field $\overline{S_\phi}$ we obtain two-parameter family of solutions of Eq. (1):

$$u(t,x) = \pm\frac{1}{3}\sqrt{12(x-\alpha)+2at+\beta},$$

where α, β are real parameters.

Remark. In order to construct new first order dynamics, we can use solutions of Eq. (7) that are different from those discussed above. There is a large set of exact solutions of the Abel equations and methods for solving them. See for example [15,16].

4. Second order dynamics of the Burgers–Huxley equation

We shall look for second order dynamics in the form of Liénard equations:

$$y'' = A(y)y' + B(y). \tag{33}$$

Then $F = y_2 - A(y_0)y_1 - B(y_0)$ and the Poisson–Lie bracket has the form

$$[\phi, F] = -A''y_1^3 - y_1^2B'' - f''y_1^2 - 2y_1\left(y_2 - \frac{1}{2}f\right)A' + B'f - f'B + 2\left(y_2 + \frac{1}{2}B\right)y_1.$$

The restriction of this bracket to Eq. (33) gives a system of ordinary differential equations with respect to the functions A, B, f:

$$\begin{cases} B'f - Bf' = 0, \\ (f - 2B)A' + 3B = 0, \\ 2A - B'' - 2AA' - f'' = 0, \\ A'' = 0. \end{cases} \tag{34}$$

Then $A(y_0) = ay_0 + b$. There are two possible cases.

If $a = 1$ then $B = -f$ and the function f is arbitrary.

If $a \neq 1$ then system (34) has non-trivial solution in case $a \neq 3/2$ only. Assuming this condition is fulfilled, we get

$$B(y_0) = \frac{af(y_0)}{2a - 3}$$

and

$$\boxed{f(y_0) = \frac{3 - 2a}{9}\left(ay_0^3 + 3by_0^2\right) + f_1y_0 + f_0} \tag{35}$$

with arbitrary constants f_0, f_1.

So, we have proved the following theorem.

Theorem 2. *The Burgers–Huxley equation has second order dynamics of Liénard's form* (33) *if either the function f has form* (35) *with $a \neq \frac{3}{2}$ or the function f is arbitrary. In the first case*

$$F = y_2 - (ay_0 + b)y_1 + \frac{a}{9(2a - 3)}\left(2a^3y_0^3 - 3(y_0 - 2b)a^2y_0^2 - 9(f_1y_0 + f_0 + by_0^2)\right),$$

and in the second one

$$F = y_2 - y_0 - b + f(y_0).$$

Remark. Note that the ordinary differential equation

$$y'' = (ay_0 + b)y' - \frac{a}{9(2a - 3)}\left(2a^3y^3 - 3(y - 2b)a^2y^2 - 9(f_1y + f_0 + by^2)\right) \tag{36}$$

has Lie algebra of shuffling symmetries with generating functions

$$\phi_1 = \overline{\phi} = (ay_0 - y_0 + b)y_1 - \frac{a(a-1)}{3}y_0^3 - b(a-1)y_0^2 + \frac{3f_1(a-1)}{2a-3}y_0 + \frac{3f_0(a-1)}{2a-3}.$$

and $\phi_2 = y_1$. This Lie algebra is commutative. According to the Lie–Bianchi theorem [4] Eq. (36) is integrable in quadratures. Below we consider two examples.

Example 2. Consider the case when $b = f_0 = f_1 = 0$. Then the Burgers–Huxley equation takes the form

$$u_t + uu_x = u_{xx} + \frac{a(3 - 2a)}{9}u^3 \tag{37}$$

and the Lienard equation is

$$y'' - ayy' + \frac{a^2}{9}y^3 = 0. \tag{38}$$

Construct the differential 1-forms

$$\omega_1 = dy_0 - y_1dx \quad \text{and} \quad \omega_2 = dy_1 + \frac{ay_0\left(ay_0^2 - 9y_1\right)}{9}dx.$$

The corresponding to the generating functions ϕ_1 and ϕ_2 vector fields are

$$\overline{S_1} = \frac{y_0}{3}(a-1)(3y_1 - ay_0^2)\frac{\partial}{\partial y_0} + \frac{a-1}{9}\left(9y_1^2 - a^2y_0^4\right)\frac{\partial}{\partial y_1},$$

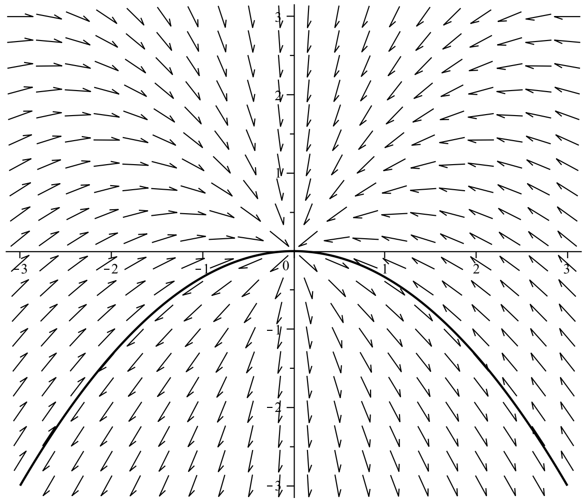

Fig. 2. The vector field $\overline{S_1}$ with $a = -1$.

$$\overline{S_2} = y_1 \frac{\partial}{\partial y_0} - \frac{a y_0}{9}(a y_0^2 - 9 y_1)\frac{\partial}{\partial y_1}$$

respectively.

The vector field $\overline{S_1}$ is shown in Fig. 2. The parabola

$$y_1 = \frac{a}{3} y_0^2 \tag{39}$$

consists of singular points of this vector field. It plays the role of a separatrix and separates the phase trajectories of two types.

Following the method described in [4,10], we compose a matrix

$$W = \begin{Vmatrix} \omega_1\left(\overline{S_1}\right) & \omega_1\left(\overline{S_2}\right) \\ \omega_2\left(\overline{S_1}\right) & \omega_2\left(\overline{S_2}\right) \end{Vmatrix} = \begin{Vmatrix} \frac{1}{3}(a-1)y_0(3y_1 - a y_0^2) & y_1 \\ (a-1)\left(y_1^2 - \frac{1}{9}a^2 y_0^4\right) & \frac{1}{9}(9y_1 - a y_0^2) a y_0 \end{Vmatrix}.$$

Determinant of the matrix is

$$\det W = \frac{1}{27}(a-1)(a y_0^2 - 3y_1)^3 \tag{40}$$

If $a y_0^2 - 3y_1 \neq 0$ and $a \neq 1$ then there exists the inverse matrix

$$W^{-1} = \begin{Vmatrix} -\dfrac{3(a y_0^2 - 9y_1) a y_0}{(a-1)(a y_0^2 - 3y_1)^3} & -\dfrac{27 y_1}{(a-1)(a y_0^2 - 3y_1)^3} \\ \dfrac{3 a y_0^2 + 9 y_1}{(a y_0^2 - 3y_1)^2} & -\dfrac{9 y_0}{(a y_0^2 - 3y_1)^2} \end{Vmatrix}.$$

Instead of the forms ω_1 and ω_2, we introduce the forms ϖ_1 and ϖ_2:

$$\begin{Vmatrix} \varpi_1 \\ \varpi_2 \end{Vmatrix} = W^{-1} \begin{Vmatrix} \omega_1 \\ \omega_2 \end{Vmatrix}.$$

Then

$$\varpi_1 = \frac{-3 a y_0 (a y_0^2 - 9y_1) dy_0 - 27 y_1 dy_1}{(a-1)(a y_0^2 - 3y_1)^3},$$

$$\varpi_2 = \frac{(-a^2 y_0^4 + 6 a y_0^2 y_1 - 9 y_1^2) dx + (3 a y_0^2 + 9 y_1) dy_0 - 9 y_0 dy_1}{(a y_0^2 - 3y_1)^2}$$

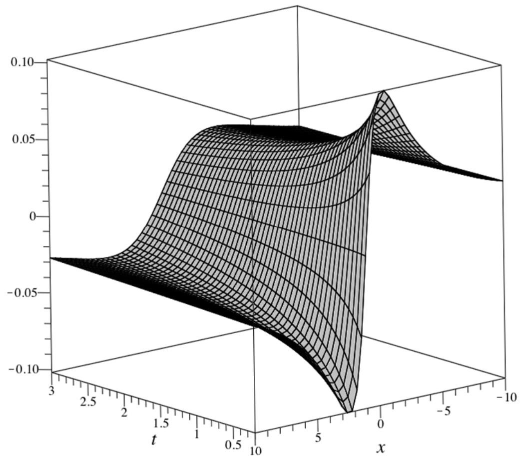

Fig. 3. The graph of solution (42) with $a = 20, C_1 = 1, C_2 = -1$.

and we get

$$\varpi_i(S_j) = \delta_{ij},$$

where δ_{ij} is the Kronecker delta.

Since the Lie bracket $[S_1, S_2] = 0$, we get

$$d\varpi_i\,(S_1, S_2) = S_1(\varpi_i(S_2)) - S_2(\varpi_i(S_1)) - \varpi([S_1, S_2]) = 0.$$

This means that the forms ϖ_1, ϖ_2 are closed.

Due to Poincare's lemma, there exist functions H_1 and H_2 such that $\varpi_1 = dH_1$ and $\varpi_2 = dH_2$. These functions are first integrals of Eq. (38). Integrating forms along an arbitrary path, we find these integrals. We do not write them here because of their bulkiness, but immediately give a general solution to Eq. (38):

$$y(x) = -\frac{6(C_1x + C_2)}{a(C_1x^2 + 2C_2x + 2)}, \tag{41}$$

where C_1, C_2 are arbitrary constants.

We get another solution of Eq. (38) if $\det W = 0$:

$$y(x) = \frac{3}{3C_1 - ax}.$$

This solution corresponds to parabola (39).

Construct solutions of Eq. (37). The translation transformation along the vector field $\overline{S_1}$ is

$$\Phi_t : \begin{cases} x \longmapsto & x, \\ y_0 \longmapsto & \dfrac{3y_0}{3 + (a-1)(ay_0^2 - 3y_1)t}, \\ y_1 \longmapsto & \dfrac{9y_1 - (a-1)(ay_0^2 - 3y_1)^2 t}{(3 + (a-1)(ay_0^2 - 3y_1)t)^2}. \end{cases}$$

Applying this transformation to solution (41), we obtain the solution of Eq. (37) (see Fig. 3):

$$u(t, x) = -\frac{6(C_1x + C_2)}{a(C_1(x^2 + 6t) + 2C_2x + 2) - 6C_1t} \tag{42}$$

Example 3. Consider the case when $a = 1, f_0 = f_1 = 0$. Then the Burgers–Huxley equation is

$$u_t + uu_x = u_{xx} + \frac{1}{9}u^3 + \frac{1}{3}bu^2. \tag{43}$$

Eq. (33) takes the form

$$y'' - yy' + \frac{1}{9}y^3 + \frac{1}{3}by^2 = 0.$$

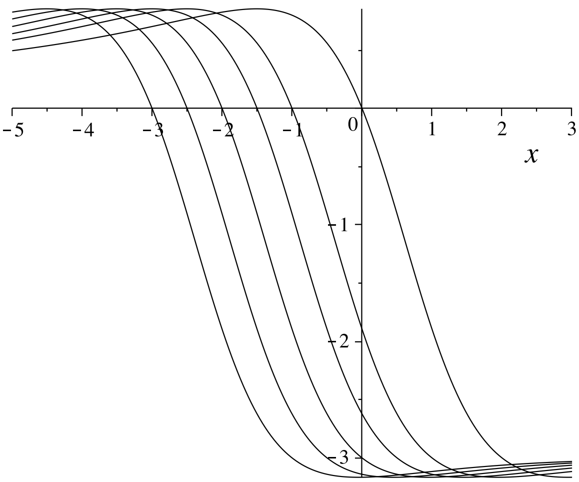

Fig. 4. Cross-sections of graph (44) for various points in time with $b = 1, C_1 = 1, C_2 = -1$.

Its general solution we get by the method described in the previous example:

$$y(x) = -\frac{3C_2 + 3b\exp(bx)}{C_1 + C_2 x + \exp(bx)}.$$

Corresponding solution of Eq. (43) is (see Fig. 4)

$$u(t,x) = -\frac{3(b\exp(b(bt+x)) + C_2)}{C_1 + C_2 bt + \exp(b(bt+x)) + C_2 x}. \tag{44}$$

Remark. Consider equation

$$u_t + \alpha u u_x = u_{xx} + f(u) \tag{45}$$

where $f \neq 0$. Direct calculations show that third order dynamics of the form

$$y''' = A(y, y')y'' + B(y, y')$$

they exist only if $\alpha = 0$. In this case Eq. (45) becomes the Kolmogorov–Petrovsky–Piskunov equation. Third-order dynamics for such equations were found in [9].

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