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Nonsymmetric Steady Shapes of Conducting Envelopes of Magnetic Stars

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Abstract—A dynamic model of a magnetized perfectly conducting envelope without internal stresses was previously proposed for the case when the envelope was maintained by a radial hypersonic stellar wind in a given gravitational field and in a given dipole magnetic field. The dynamic equilibrium problem for an axisymmetric rotating envelope is considered on the basis of this model. The surface density and the angular velocity are assumed to be constant. The edge-free solutions nonsymmetric with respect to the equatorial plane are found. The bottle-like shapes going to infinity are studied numerically.

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A dynamic model of a magnetized perfectly conducting envelope without internal stresses was previously proposed in [1] for the case when the envelope was maintained by a radial hypersonic stellar wind in a given gravitational field and in a given dipole magnetic field. The dynamic equilibrium problem for an axisymmetric rotating envelope is considered on the basis of this model. The surface density and the angular velocity are assumed to be constant. The problem can be reduced to the analytic and numerical integration of an ordinary differential equation. The exact solutions nonsymmetric with respect to the equatorial plane are found; these solutions do not have the edges typical for the majority of symmetric envelopes [1]. The nonsymmetric bottle-like shapes of envelopes going to infinity are studied numerically.

1. INTRODUCTION

The photographs of planetary nebulae recently obtained with the use of high-resolution telescopes reveal the existence of various multilayer structures consisting of gas-dust envelopes near stars. The Cat's Eye nebula (NGC 6543) is a typical example. Its photograph [2] distinctly shows the two bottle-like shapes, which are oppositely directed and are surrounded by a number of spherical envelopes.

The traditional theory of flat plasma disks [3] is used to study astrophysical objects with sufficiently strong magnetic fields. In addition to this theory, also of interest is the study of steady spatial configurations of thin conducting envelopes. A number of theoretical studies are devoted to such objects whose evolution may be accompanied even by cumulative effects [4, 5]. A stellar wind effect should also be taken into account, since this effect may become dominant.

2. THE EQUATIONS OF ENVELOPE MOTION

The equations describing the motion of an envelope in the central gravitational field of a star are of the form

$$\sigma \mathbf{r}_{tt} = k\rho w_n \mathbf{w} - \frac{\sigma G m \mathbf{r}}{r^3} + \frac{\mathbf{j} \times \mathbf{H}_e}{c}, \tag{1}$$

where m is the mass of the star, $\mathbf{H}_e$ is the external magnetic field, $\mathbf{r}$ is the radius vector of envelope particles, σ is the envelope surface density, ρ is the radial stellar wind density, $\mathbf{w}$ is its velocity, w_n is the velocity projection on the normal to the envelope, $\mathbf{j}$ is the density of the surface current, G is the gravitation constant, c is the speed of light, and k is the absorption coefficient for the stellar wind momentum flux. It is assumed that the absorption of the stellar wind mass flux is absent.

In the spherical coordinates r, θ, and φ, the steady stellar wind distribution is specified by the relations

$$r^2 \rho w = \frac{Q}{4\pi}, \qquad w = \left(w_\infty^2 + \frac{2Gm}{r} \right)^{1/2}, \tag{2}$$

where Q is the strength of the wind source. The coordinates r and θ are related to the cylindrical coordinates r_c and z by the formulas $r_c = r\sin\theta$ and $z = r\cos\theta$.

The field of an external magnetic dipole of strength M has the components

$$H_e^r = \frac{2M\cos\theta}{r^3}, \qquad H_e^\theta = \frac{M\sin\theta}{r^4}. \tag{3}$$

Equation (1) should be supplemented by the known formulas for the mass density σ and for the surface current $\mathbf{j}$; these formulas contain the components of the surface metric tensor, the initial density, and the jump of the self-magnetic field frozen in the envelope [1]. In the case of the steady rotation problem for an envelope, however, these relations can always be fulfilled by selecting the initial data. Thus, the density and the current can be regarded as independent quantities.

3. THE DYNAMIC EQUILIBRIUM OF AN ENVELOPE

When an envelope rotates steadily, for the acceleration we have $\mathbf{r}_{tt} = -\omega^2\mathbf{r}_c$, where ω is the angular velocity and $\mathbf{r}_c$ is the cylindrical radius. Projecting Eq. (1) onto the r- and θ-axes, taking into account (2) and (3), and eliminating the surface current, we come to the equations

$$\frac{3\omega^2 r}{2}\sin^2\theta + \frac{kQw(r)}{4\pi\sigma r\sqrt{r^2 + r'^2(\theta)}} = \frac{Gm}{r^2}, \tag{4}$$

$$\frac{j_\varphi}{r\sin\theta} = -\frac{r^4\sin\theta\omega^2\sigma c}{2M}. \tag{5}$$

The latter relation represents the physical component of the current density.

Note that Eq. (4) written with the use of $\theta'(r)$ has the solution $\theta = \text{const}$; in other words, this equation describes the differential rotation of a cone with $\omega = \omega(r)$ (in particular, the rotation of a flat disk [3]) in the absence of the stellar wind effect. Below we consider some other cases.

Thus, Eqs. (4) and (5) relate the four functions r, ω, σ, and j_φ of the angular variable θ; these functions are subject to a number of inequalities [1]. In order to make the problem more specific, we assume that ω and σ are constant; in addition, we nondimensionalize Eq. (4), which specifies the envelope shape.

Let us introduce the parameters

$$R = \frac{2Gm}{w_\infty^2}, \qquad \varepsilon = \left(\frac{kQw_\infty}{4\pi Gm\sigma}\right)^2, \qquad \delta = \frac{12\omega^2G^2m^2}{w_\infty^6}$$

and the dimensionless variable $r_1 = r/R$. Dropping the subscript from the dimensionless radius, we get

$$r'(\theta) = \pm\frac{\left(\varepsilon(r^2 + r) - r^2(1 - \delta r^3\sin^2\theta)^2\right)^{1/2}}{1 - \delta r^3\sin^2\theta}. \tag{6}$$

By virtue of (4), the denominator of (6) is positive. Moreover, the radicand should be nonnegative.

4. THE EXACT SOLUTIONS

First we discuss the following limiting cases: (i) the stellar wind is absent ($\varepsilon = 0$) and (ii) the magnetic field and the rotation are absent ($\delta = 0$).

When $\varepsilon = 0$, from (6) we get

$$1 = \delta r^3\sin^2\theta, \qquad \text{or} \qquad z^2 = 1/(\delta^2 r_c^4) - r_c^2.$$

This case corresponds to an everywhere (except for $r_c = 0$) smooth double "bottle" expanding at the equator. This figure is symmetric with respect to the plane $z = 0$.

When $\delta = 0$, there are possible the following cases (the solutions are given for the plus sign; otherwise, the variable θ should be replaced by $\pi - \theta$):

(i) for $\varepsilon > 1$

$$\frac{2}{\sqrt{\varepsilon - 1}}\ln\frac{\sqrt{r} + \sqrt{r + \varepsilon/(\varepsilon - 1)}}{\sqrt{r_0} + \sqrt{r_0 + \varepsilon/(\varepsilon - 1)}} = \theta;$$

(ii) for $\varepsilon = 1$

$$2\left(\sqrt{r} - \sqrt{r_0}\right) = \theta;$$

(iii) for $\varepsilon < 1$

$$\frac{2}{\sqrt{1-\varepsilon}}\left(\arcsin\frac{\sqrt{r}}{\sqrt{\varepsilon/(1-\varepsilon)}} - \arcsin\frac{\sqrt{r_0}}{\sqrt{\varepsilon/(1-\varepsilon)}}\right) = \theta.$$

Here r_0 is the radius of the envelope for $\theta = 0$.

In each of these cases, we can construct a figure symmetric with respect to the plane $z = 0$; at the equator ($\theta = \pi/2$), however, an edge is always observed. In addition, the cone vertices or the depressions are situated at the poles. However, the above local relations can be used to continue the solution smoothly through the equator, which is more acceptable from the standpoint of stability of the solution (we mean the instability of the edge of the Laplace protosolar disk in connection with the planet formation hypothesis [3, 4]). In this case, we obtain the shapes nonsymmetric with respect to the reflection z. The corresponding shapes are illustrated in Figs. 1a–1c for $\varepsilon = 0.5$, $\varepsilon = 1$, and $\varepsilon = 2$, respectively. When $\varepsilon < 1$, all envelopes are bounded by the sphere $r = \varepsilon/(1-\varepsilon)$ and some of the envelopes are not closed; when $\varepsilon \geq 1$, all envelopes are closed.

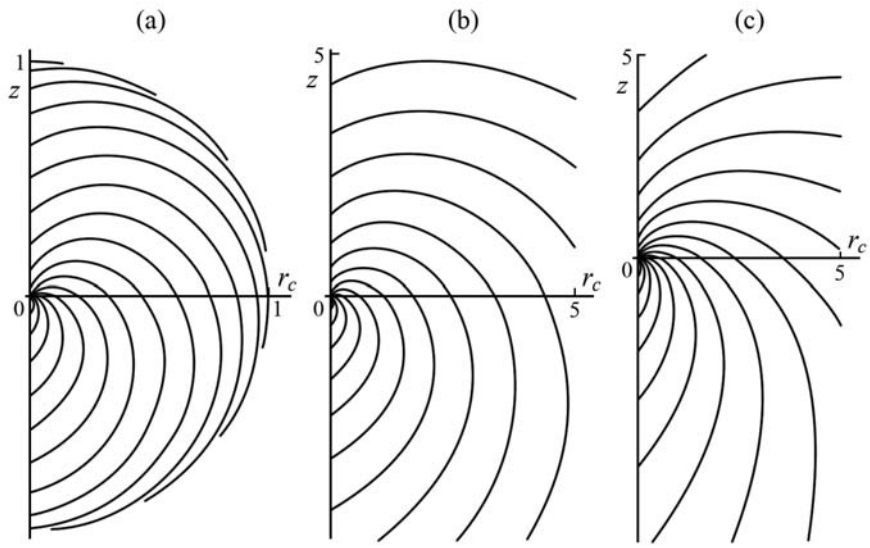

Fig. 1.

A perfectly smooth shape can be obtained only in the case of spherical symmetry when $r = \varepsilon/(1-\varepsilon)$ if $\varepsilon < 1$. The last inequality can be used to estimate ε from observations. In particular, for $r \gg 1$ we may assume that $\varepsilon = 1$. In this case we come to a power law relating the quantities in the expression for ε. In terms of dimensional quantities, we have the following estimates: $m \sim 10^{33}$ g, the nebula mass μ is equal to 10^{32} g, $r \sim 10^{17}$ cm, and $w_\infty \sim 10^7$ cm/s [2]. Then, $R \sim 10^{11}$ cm, $r/R \sim 10^6$, $\sigma = \mu/(4\pi r^2) \sim 10^{-3}$ g/cm^2, and $Q \approx 4\pi G m\sigma/w_\infty \approx 4\pi\rho r^2 w_\infty$. From here we obtain $\rho \approx Gm\sigma/(r^2 w_\infty^2) \sim 10^{-25}$ g/cm^3 (this estimate approximately corresponds to the stellar wind density in the solar system). Usually, δ is very small but can be essential when the stellar wind is weak, i.e., when $w_\infty \sim 10^8\,\omega^{1/3}$ (in the CGS system of units).

5. BOTTLE-LIKE SHAPES

In the general case when $\delta \neq 0$ and $\varepsilon \neq 0$, Eq. (6) has the following peculiarity: in addition to the bounded axisymmetric closed and open envelopes, there appear unbounded bottle-shaped envelopes corresponding to the separatrix that goes to infinity as $\theta \to 0$ or $\theta \to \pi$.

There are possible three qualitatively different cases. These cases are illustrated in Fig. 2 by the numerical solutions to Eq. (6) on the plane (r_c, z) for the minus sign (for the plus sign, the "bottleneck" rotates in the opposite direction; the boundaries of the existence domain for the solutions are shown by the dotted lines). In the first case (Fig. 2a; $\delta = 0.15$ and $\varepsilon = 0.5$), on the plane (r, θ) the separatrix continues onto the interval $\theta \in (0; \pi)$ and the coordinate origin is situated inside the "bottle"; the conical depression is observed at the

bottom of the "bottle". In the second case (Fig. 2b; $\delta = 0.15$ and $\varepsilon = 1.1$), on the plane (r, θ) the separatrix intersects the θ-axis for $\theta < \pi$, so that the bottle-shaped envelope issues from the coordinate origin. In the third case (Fig. 2c; $\delta = 0.03$ and $\varepsilon = 0.5$), the separatrix reaches the boundary of the existence domain and the "bottle" turns out to be without a "bottom".

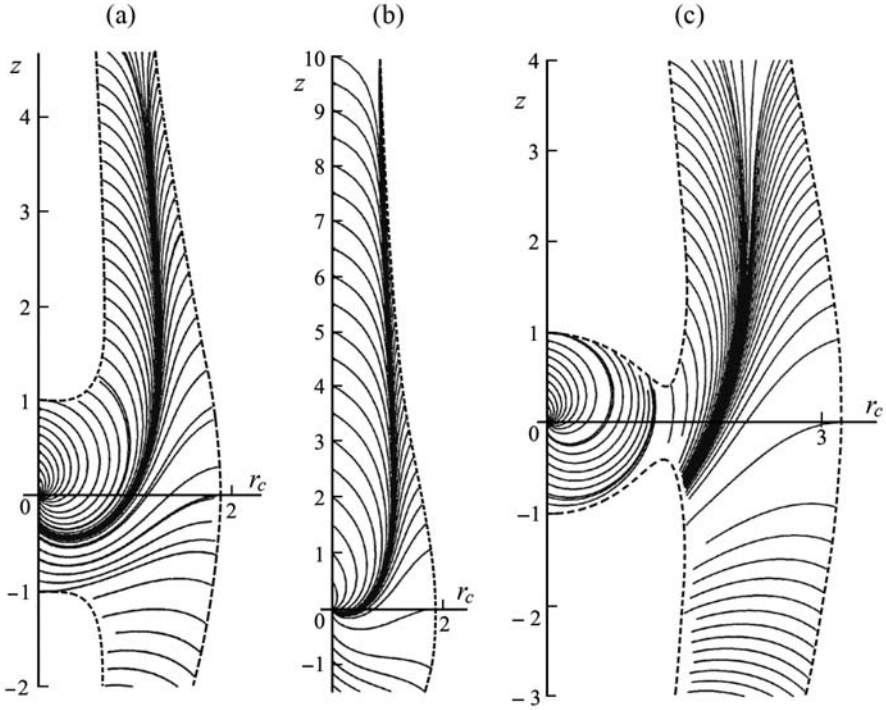

Fig. 2.

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