Carleson measures and extensions of some classical inequalities

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Abstract

It is observed that due to Carleson's harmonic extension theorem some classical inequalities, such as those of Hilbert and Hausdorff-Young can be stated in a stronger "maximal" form. For instance, if $f \in L^p(\mathbb{R})$, $p > 1$, and $g = Hf$ is its conjugate function, then the function

$$g^*(r) = \sup_{0<\theta<\pi} g(re^{i\theta})$$

belongs to $L^p(\mathbb{R}_+)$.

Abstracting the essence of a proof, we are led to a measure-theoretic construction involving the notions of conditional essential supremum and conditional Carleson measure.

Notation

Playground:

- $\mathbb{H}$ upper half-plane
- $L^p(\mathbb{R})$, $L^p(\mathbb{R}_+)$ (These will be shorthanded as L^p)
- $L^p(\mu)$ ($\mu \in \mathcal{B}(\mathbb{H})$)
- Weak spaces $L^{1,\infty}(\mu)$

Players:

- Harmonic extension (Poisson) $v(x+iy) = \mathcal{P}_y * u(x)$
- Cauchy operator $v(z) = \int (z-t)^{-1} u(t)\, dt$
- Hilbert transform $v = PV(1/x) * u$
- Fourier-Laplace transform $v(z) = \int_0^\infty u(t) e^{itz}\, dt$

Questions

I. Admissible weights (in target domain)

- Let $T : u(x) \to v(z)$, $z \in \mathbb{H}$.
- For which measures do we have $\|v\|_{L^s(\mu)} \leq C\|u\|_{L^p}$?
  ($s = p' \equiv (1-p^{-1})^{-1}$ for the Fourier-Laplace and $s = p$ in other cases).

II. Admissible maximizations

- Given a family S_α, $\alpha \in \mathcal{A}$, of "slices", define

$$T^* u(\alpha) \equiv v^*(\alpha) = \sup_{z \in S_\alpha} |v(z)|.$$

Carleson's Theorem

Answer to the weight question in case of T the Poisson operator (harmonic extension)

Theorem

TFAE:

- T *bounded from* $L^p(\mathbb{R})$ *to* $L^p(\mu)$ *for some* $p > 1$;
- T *bounded from* $L^p(\mathbb{R})$ *to* $L^p(\mu)$ *for all* $p > 1$;
- T *bounded from* $L^1(\mathbb{R})$ *to* $L^{1,\infty}(\mu)$;
- $\|\mu\|_{\mathcal{C}} = \sup_B |\mu(B)|/|\ell(B)| < \infty$,
  B — *Carleson boxes*

Definition

Let $\mu \in \mathcal{B}(\overline{\mathbb{H}})$. We say that $\mu \in \mathrm{HY}(p)$ (μ *has the Hausdorff-Young property of order* p) if

$$\|v(z)\|_{L^{p'}(\mu)} \leq C\|u\|_{L^p(\mathbb{R}_+)},$$

where v is the Fourier-Laplace transform of u.

If $p > 1$ and C is the smallest possible, we call $N_{\mathcal{HY},p}(\mu) = C^{p'}$, the *Hausdorff-Young norm of order* p of μ.

If $p = 1$, the norm $N_{\mathcal{HY},1}(\mu)$ is not defined.

Hausdorff-Young meets Carleson

Answer to the weight question in case of T the Fourier-Laplace transform

Theorem (HY-characterization of Carleson measures)

TFAE:

- *(1)* $\mu \in \mathrm{HY}(p)$ *for some* $p \in (1, 2]$.
- *(2)* $\mu \in \mathrm{HY}(p)$ *for all* $p \in [1, 2]$.
- *(3)* μ *is a Carleson measure on* $\overline{\mathbb{H}}$.

Proof.

- (1)⇒(3): by testing on appropriate functions
- (3)⇒(2): by factorization (Hausdorff-Young, then Carleson):
  Fourier transform $u \to \hat{u}$, $\|\hat{u}\|_{L^{p'}(\mathbb{R})} \le C\|u\|_{L^p(\mathbb{R}_+)}$
  Harmonic extension $\hat{u} \to v$; $\|v\|_{L^{p'}(\overline{\mathbb{H}},\mu)} \le C\|\hat{u}\|_{L^{p'}(\mathbb{R})}$.

□

Attention to constants!

Hausdorff-Young inequality

$$\|\hat{u}\|_{L^{p'}(\mathbb{R})} \leq B(p) \|u\|_{L^p(\mathbb{R})}.$$

Best constant [Babenko 1961, Beckner 1974]

$$B(p) = (2\pi)^{1/p'} \left(\frac{p^{1/p}}{p'^{1/p'}} \right)^{1/2}$$

Order-best constant in Carleson's thm (v is the harmonic extension of u)

$$\|v\|^p_{L^p(d\mu)} \leq C \max(1, p') \|\mu\|_{\mathcal{C}} \|u\|^p_{L^p(\mathbb{R})},$$

where C is an absolute constant.

Hausdorff-Young-Carleson with constants

Theorem (HY-characterization of Carleson measures, quantitative version)

For $1 < p \le 2$ and some absolute constants C_1, C_2 we have

$$C_1(p-1)\|\mu\|_{\mathcal{C}} \le N_{\mathcal{HY},p}(\mu) \le C_2\sqrt{p-1}\,\|\mu\|_{\mathcal{C}}$$

The bounds are order-sharp: the coefficient in l.h.s. cannot be made $\gg p-1$ and coefficient in r.h.s. cannot be made $\ll p-1$.

Definition

Scale of non-equivalent norms for Carleson measures:

$$N_{\mathcal{HY},(1,\alpha)}(\mu) := \limsup_{p\to 1^+}(p-1)^{-\alpha}N_{\mathcal{HY},p}(\mu), \qquad \frac{1}{2} \le \alpha \le 1.$$

The classes $\mathrm{HY}(1,\alpha) = \{\mu \in N_{\mathcal{C}}(\mathbb{C}_+) \, : \, N_{\mathcal{HY},(1,\alpha)}(\mu) < \infty\}$ expand as α decreases from 1 to 1/2.

Open question

Characterize membership in the classes $\mathrm{HY}(1, \alpha)$ in terms of a geometric property of the Carleson measure μ.

Example

- $\mu = dx \otimes \delta(y)$ is in $\mathrm{HY}(1, 1/2)$ but not in any $\mathrm{HY}(1, 1/2 - \varepsilon)$.
- $\mu = \delta(x) \otimes dy$ is in $\mathrm{HY}(1, 1)$.

Maximal inequalities: motivation

Theorem

For $v(z)$ the Fourier-Laplace transform of u let

$$v^*(y) = \sup_x |v(x+iy)|.$$

Then for $1 < p \leq 2$

$$\|v^*\|_{L^{p'}(\mathbb{R}_+)} \leq C\|u\|_{L^p(\mathbb{R}_+)}.$$

Proof.

Worst case: $u \geq 0$, $x = 0$. Then $v^*(y) = v(y) = \int_0^\infty e^{-ty} u(t)\, dt$, and Theorem follows from Hardy's (1933) adaptation of the Hausdorff-Young inequality to the Laplace-transform. □

Maximal angular inequalities

For $v(z)$ defined in $\mathbb{H}$, let

$$v^*(r) = \sup_{0<\theta<\pi} |v(re^{i\theta})|.$$

Theorem (Max.ang.thm for F-L transform)

Let $v(z)$ be the Fourier-Laplace transform of u. Then

$$\|v^*\|_{L^{p'}(\mathbb{R}_+)} \le C\|u\|_{L^p(\mathbb{R}_+)} \qquad (1 < p \le 2).$$

Theorem (Max.ang.thm for harmonic extension)

For $v(z)$ the harmonic extension of u and v^* as above we have

$$\|v^*\|_{L^p(\mathbb{R}_+)} \le C\|u\|_{L^p(\mathbb{R})} \qquad (1 < p < \infty)$$

and the $(1,1)$-weak inequality

$$\|v^*\|_{L^{1,\infty}(\mathbb{R}_+)} \le C\|u\|_{L^1(\mathbb{R})}.$$

Sketch of proof

Let $S(r) = \{z \in \mathbb{H} : \ |z| = R\}$ (semicircle used in def. of $v^*(r)$).
Suppose $v(z)$ is defined in $\mathbb{H}$. Fix $\varepsilon > 0$. Let

$$\Omega(r) = \{z \in S(r) : \ |v(z)| > (1-\varepsilon)v^*(r).$$

Let $d\omega_r$ be the Euclidean measure on $S(r)$, so that
$dx \otimes dy = dr \otimes d\omega_r$. Define the density function

$$\rho(r) = \begin{cases} 1/\omega_r(Y(r)), & \text{if} \quad \omega_r(Y(r)) \neq 0 \\ 0, & \text{otherwise.} \end{cases}$$

and the measure $\mu = \rho(r)\, dr \otimes d\omega_r$.
This construction yields: $\|v^*\|_{L^p(\mathbb{R}_+)} \leq (1-\varepsilon)^{-1}\|v\|_{L^p(\mathbb{H},\mu)}$.
Technical step: μ is a Carleson measure with norm ≤ 2.
Application of Carleson's thm and $\varepsilon \to 0$ concludes the proof.

Other maximal functions

Same results hold true for v^* defined in a different way. The measure space $(\mathbb{R}_+, dr)$ should be adapted appropriately.

Examples:

- $v^*(x) = \sup_{y>0} |v(x+iy)|$; compute $\|v^*\|_p$ wrt $(\mathbb{R}, dx)$
- $v_n^* = \sup\limits_{n<x<n+1} \sup\limits_{y>0} |v(x+iy)|$; here $\|v^*\|_p$ is the norm in the sequence space.

Maximal H-Y inequality, general discrete weights

$K(t,R)$ denotes the semicircle in $\mathbb{H}$ of radius R centered at t.

Theorem

Suppose a sequence of sets $\Omega_n \subset \overline{\mathbb{H}}$ and positive weights w_n satisfies the property

$$\forall R > 0, t \in \mathbb{R} \qquad \sum_{n\colon\, \Omega_n \cap K(t,R) \neq \emptyset} w_n \leq AR,$$

where A is independent of R (but depends on Ω_n and w_n). Then for all $p \in [1,2]$

$$\left(\sum_n w_n \sup_{\Omega_n} |v(z)|^{p'} \right)^{1/p'} \leq CA^{1/p} \|u\|_p,$$

where C is an absolute constant and $v(z)$ is the Fourier-Laplace transform of u.

A look behind the scenes

- Identify the sequence $v_n^* = \sup_{\Omega_n} |v(z)|$ as *conditional supremum* of v with respect to a σ-algebra $\mathcal{G}$ generated by the collection of sets Ω_n.
- Call the sequence of weights satisfying the property

$$\forall R > 0, t \in \mathbb{R} \qquad \sum_{n:\, \Omega_n \cap K(t,R) \neq \emptyset} w_n \leq AR,$$

  the conditional (wrt $\mathcal{G}$) Carleson condition.
- Details, continuous weights ... technicalities

Discrete analogue:

Theorem

Let

$$b_n = \sum_{m=-\infty}^{\infty} \frac{a_m}{m + n + 1/2}.$$

Then for all $p \in (1, \infty)$

$$\|b\|_{\ell_p} \leq C_p \|a\|_{\ell_p}.$$

Discrete analogue:

Theorem

Let

$$b_n = \sup_\theta \left| \sum_{m=-\infty}^{\infty} \frac{a_m}{me^{i\theta} + n + 1/2} \right| .$$

Then for all $p \in (1, \infty)$

$$\|b\|_{\ell_p} \le C_p \|a\|_{\ell_p} .$$

Discrete analogue:

Theorem

Let

$$b_n = \sup_\theta \left| \sum_{m=-\infty}^{\infty} \frac{a_m}{me^{i\theta} + n + 1/2} \right| .$$

Then for all $p \in (1, \infty)$

$$\|b\|_{\ell_p} \leq C_p \|a\|_{\ell_p} .$$

Thus, the Hilbert inequality works for a matrix
$h_{mn} = (me^{i\theta_n} + n + 1/2)^{-1}$ with any choice of θ_n's.

Variations

Maximal Hilbert transform

Theorem

Let

$$g^*(x) = \sup_{\varepsilon>0} \int_{|x-y|>\varepsilon} \frac{f(y)\,dy}{x-y}$$

Then for all $p \in (1, \infty)$

$$\|g^*\|_p \leq C_p \|f\|_p.$$

Variations

Max-Maximal Hilbert transform

Conjecture

Let

$$g^{**}(x) = \sup_{\varepsilon>0} \sup_{0<\theta<\pi} \int_{|xe^{i\theta}-y|>\varepsilon} \frac{f(y)\,dy}{xe^{i\theta}-y}$$

Then for all $p \in (1,\infty)$

$$\|g^{**}\|_p \le C_p \|f\|_p.$$

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