

Градиент функции в произвольных координатах.

$$\text{grad } f = \frac{\partial f}{\partial x} \vec{i} + \frac{\partial f}{\partial y} \vec{j} + \frac{\partial f}{\partial z} \vec{k}$$

Пусть $x = x(\alpha, \beta, \gamma)$

$$y = y(\alpha, \beta, \gamma)$$

$$z = z(\alpha, \beta, \gamma), \text{ где } \alpha, \beta, \gamma -$$

— новые координаты

$$\left\{ \begin{aligned} \frac{\partial f}{\partial \alpha} &= \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial \alpha} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial \alpha} + \frac{\partial f}{\partial z} \cdot \frac{\partial z}{\partial \alpha} \\ \frac{\partial f}{\partial \beta} &= \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial \beta} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial \beta} + \frac{\partial f}{\partial z} \cdot \frac{\partial z}{\partial \beta} \\ \frac{\partial f}{\partial \gamma} &= \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial \gamma} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial \gamma} + \frac{\partial f}{\partial z} \cdot \frac{\partial z}{\partial \gamma} \end{aligned} \right.$$

$$\vec{r} = x \vec{i} + y \vec{j} + z \vec{k}$$

$$\vec{e}_\alpha = \frac{\partial \vec{r}}{\partial \alpha} / \left| \frac{\partial \vec{r}}{\partial \alpha} \right|$$

$$\text{T.e. } \vec{e}_\alpha = \frac{1}{\sqrt{x_\alpha^2 + y_\alpha^2 + z_\alpha^2}} (x_\alpha \vec{i} + y_\alpha \vec{j} + z_\alpha \vec{k})$$

B transformation matrix:

$$\begin{pmatrix} \varphi_x \\ \varphi_y \\ \varphi_z \end{pmatrix} = \begin{pmatrix} x_x & y_x & z_x \\ x_y & y_y & z_y \\ x_z & y_z & z_z \end{pmatrix} \begin{pmatrix} \varphi_x \\ \varphi_y \\ \varphi_z \end{pmatrix}$$

$$\begin{pmatrix} \vec{e}_x \\ \vec{e}_y \\ \vec{e}_z \end{pmatrix} = \begin{pmatrix} x_x/|\vec{e}_x| & y_x/|\vec{e}_x| & z_x/|\vec{e}_x| \\ x_y/|\vec{e}_y| & y_y/|\vec{e}_y| & z_y/|\vec{e}_y| \\ x_z/|\vec{e}_z| & y_z/|\vec{e}_z| & z_z/|\vec{e}_z| \end{pmatrix} \times \begin{pmatrix} \vec{i} \\ \vec{j} \\ \vec{k} \end{pmatrix}$$

Signature $A = \begin{pmatrix} x_x & y_x & z_x \\ x_y & y_y & z_y \\ x_z & y_z & z_z \end{pmatrix}$

$$\tilde{A} = \begin{pmatrix} x_x/|\vec{e}_x| & y_x/|\vec{e}_x| & z_x/|\vec{e}_x| \\ x_y/|\vec{e}_y| & y_y/|\vec{e}_y| & z_y/|\vec{e}_y| \\ x_z/|\vec{e}_z| & y_z/|\vec{e}_z| & z_z/|\vec{e}_z| \end{pmatrix}$$

Normalized gradient:

$$\text{grad } f = (f_x, f_y, f_z) \begin{pmatrix} \vec{i} \\ \vec{j} \\ \vec{k} \end{pmatrix}$$

$$\begin{pmatrix} \varphi_x & \varphi_y & \varphi_z \end{pmatrix} = \begin{pmatrix} \varphi_x & \varphi_y & \varphi_z \end{pmatrix} \begin{pmatrix} x_x & x_y & x_z \\ y_x & y_y & y_z \\ z_x & z_y & z_z \end{pmatrix} = A^T$$

Тогда $\text{grad } f = (f_x, f_y, f_z)$

$$\cdot (A^T)^{-1} \cdot (\tilde{A}^{-1}) \begin{pmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \\ \frac{\partial f}{\partial z} \end{pmatrix}$$

Пример: сферические координаты

$$\begin{cases} x = r \sin \theta \cos \varphi \\ y = r \sin \theta \sin \varphi \\ z = r \cos \theta \end{cases}$$

Обозначим: $\alpha = r$; $\beta = \theta$; $\gamma = \varphi$

$$\begin{cases} x_\alpha = \sin \beta \cos \gamma \\ y_\alpha = \sin \beta \sin \gamma \\ z_\alpha = \cos \beta \end{cases}$$

$$\begin{cases} x_\beta = -r \sin \beta \sin \gamma \\ y_\beta = r \sin \beta \cos \gamma \\ z_\beta = 0 \end{cases}$$

$$\begin{cases} x_\gamma = -r \cos \beta \sin \gamma \\ y_\gamma = r \cos \beta \cos \gamma \\ z_\gamma = -r \sin \beta \end{cases}$$

$$A = \begin{pmatrix} \sin \theta \cos \varphi & \sin \theta \sin \varphi & \cos \theta \\ z \cos \theta \cos \varphi & z \cos \theta \sin \varphi & -z \sin \theta \\ -z \sin \theta \sin \varphi & z \sin \theta \cos \varphi & 0 \end{pmatrix}$$

$$\det A = (-z \sin \theta \sin \varphi) - (-z \sin^2 \theta \sin \varphi -$$

$$- z \cos^2 \theta \sin \varphi) - z \sin \theta \cos \varphi \cdot$$

$$\cdot (-z \sin^2 \theta \cos \varphi - z \cos^2 \theta \cos \varphi) =$$

$$= z^2 \sin \theta \sin^2 \varphi \rightarrow z^2 \sin \theta \cos^2 \varphi = z^2 \sin \theta$$

$$A^{-1} = \frac{1}{|\det A|} \times \begin{pmatrix} z^2 \sin^2 \theta \cos \varphi & z^2 \sin^2 \theta \sin \varphi & z^2 \sin \theta \cos \theta \\ z \sin \theta \cos \theta \cos \varphi & z \sin \theta \cos \theta \sin \varphi & -z \sin \theta \\ -z \sin \varphi & +z \cos \varphi & 0 \end{pmatrix}^T =$$

$$= \begin{pmatrix} \sin \theta \cos \varphi & \sin \theta \sin \varphi & \cos \theta \\ \frac{1}{z} \cos \theta \cos \varphi & \frac{1}{z} \cos \theta \sin \varphi & -\frac{1}{z} \sin \theta \\ \left(-\frac{1}{z}\right) \frac{\sin \varphi}{\sin \theta} & \left(\frac{1}{z}\right) \frac{\cos \varphi}{\sin \theta} & 0 \end{pmatrix}^T =$$

$$= \begin{pmatrix} \sin \theta \cos \varphi & \frac{1}{z} \cos \theta \cos \varphi & \left(-\frac{1}{z}\right) \frac{\sin \varphi}{\sin \theta} \\ \sin \theta \sin \varphi & \frac{1}{z} \cos \theta \sin \varphi & \left(-\frac{1}{z}\right) \frac{\cos \varphi}{\sin \theta} \\ \cos \theta & \left(-\frac{1}{z}\right) \sin \theta & 0 \end{pmatrix}$$

$$\tilde{A} = \begin{pmatrix} x_a/|\vec{\Sigma}_a| & y_a/|\vec{\Sigma}_a| & z_a/|\vec{\Sigma}_a| \\ x_B/|\vec{\Sigma}_B| & y_B/|\vec{\Sigma}_B| & z_B/|\vec{\Sigma}_B| \\ x_\varphi/|\vec{\Sigma}_\varphi| & y_\varphi/|\vec{\Sigma}_\varphi| & z_\varphi/|\vec{\Sigma}_\varphi| \end{pmatrix}$$

$$|\vec{\Sigma}_a| = \sqrt{x_a^2 + y_a^2 + z_a^2}$$

$$|\vec{\Sigma}_a| = \sqrt{\sin^2\theta \cos^2\varphi + \sin^2\theta \sin^2\varphi + \cos^2\theta} = 1$$

$$|\vec{\Sigma}_B| = \sqrt{r^2 \cos^2\theta \cos^2\varphi + r^2 \cos^2\theta \sin^2\varphi + r^2 \sin^2\theta} = r$$

$$|\vec{\Sigma}_\varphi| = \sqrt{r^2 \sin^2\theta \sin^2\varphi + r^2 \sin^2\theta \cos^2\varphi} = r \sin\theta$$

$$\tilde{A} = \begin{pmatrix} \sin\theta \cos\varphi & \sin\theta \sin\varphi & \cos\theta \\ \cos\theta \cos\varphi & \cos\theta \sin\varphi & -\sin\theta \\ -\sin\varphi & \cos\varphi & 0 \end{pmatrix}$$

$$\det \tilde{A} = (-\sin\varphi)(-\sin^2\theta \sin\varphi - \cos^2\theta \sin\varphi) - \cos\varphi(-\sin^2\theta \cos\varphi - \cos^2\theta \cos\varphi) = 1$$

$$\begin{aligned}
 \tilde{A}^{-1} &= \begin{pmatrix} \sin \theta \cos \varphi & \sin \varphi \sin \theta & \cos \theta \\ \cos \varphi \cos \theta & \sin \varphi \cos \theta & -\sin \theta \\ -\sin \varphi & \cos \varphi & 0 \end{pmatrix} \\
 &= \begin{pmatrix} \sin \theta \cos \varphi & \cos \varphi \cos \theta & -\sin \varphi \\ \sin \varphi \sin \theta & \sin \varphi \cos \theta & \cos \varphi \\ \cos \theta & -\sin \theta & 0 \end{pmatrix} \\
 &= \text{Proposed: } \tilde{A} \tilde{A}^{-1} = \begin{pmatrix} \sin \theta \cos \varphi & \sin \theta \sin \varphi & \cos \theta \\ \cos \theta \cos \varphi & \cos \theta \sin \varphi & -\sin \theta \\ -\sin \varphi & \cos \varphi & 0 \end{pmatrix} \\
 &\quad \times \begin{pmatrix} \sin \theta \cos \varphi & \cos \varphi \cos \theta & -\sin \varphi \\ \sin \varphi \sin \theta & \sin \varphi \cos \theta & \cos \varphi \\ \cos \theta & -\sin \theta & 0 \end{pmatrix} \\
 &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}
 \end{aligned}$$

Task: $\text{grad } f = \left(\frac{\partial f}{\partial r}, \frac{\partial f}{\partial \theta}, \frac{\partial f}{\partial \varphi} \right)^T$

$$\times (A^T)^{-1} \times (\tilde{A})^{-1} \begin{pmatrix} \vec{e}_r \\ \vec{e}_\theta \\ \vec{e}_\varphi \end{pmatrix}$$

$$(A^T)^T \times (\tilde{A})^{-1} = \begin{pmatrix} \sin \theta \cos \varphi & \sin \theta \sin \varphi & \cos \theta \\ \frac{1}{2} \cos \theta \cos \varphi & \frac{1}{2} \cos \theta \sin \varphi & -\frac{1}{2} \sin \theta \\ \left(-\frac{1}{2}\right) \frac{\sin \varphi}{\sin \theta} & \left(\frac{1}{2}\right) \frac{\cos \varphi}{\sin \theta} & 0 \end{pmatrix}$$

$$\times \begin{pmatrix} \sin \theta \cos \varphi & \cos \varphi \cos \theta & -\sin \varphi \\ \sin \varphi \sin \theta & \sin \varphi \cos \theta & \cos \varphi \\ \cos \theta & -\sin \theta & 0 \end{pmatrix} =$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & (1/2) & 0 \\ 0 & 0 & (1/2 \sin \theta) \end{pmatrix}$$

Empereur Cramer's

1

1

$$\sin^2 \theta \cos^2 \varphi + \sin^2 \theta \sin^2 \varphi + \cos^2 \theta = 1$$

1

2

$$\sin \theta \cos \varphi \times \cos \varphi \cos \theta + \sin \theta \sin \varphi \sin \varphi \cos \theta + \cos \theta (-\sin \theta) = 0$$

1

3

$$\sin \theta \cos \varphi (-\sin \varphi) + \sin \theta \sin \varphi \cos \varphi = 0$$

$\frac{1}{2}$ $\frac{1}{2}$ $\left(\frac{1}{2}\right) \cos \theta \cos \varphi \sin \theta \cos \varphi + \left(\frac{1}{2}\right) \cos \theta \sin \varphi$
 $\times \sin \varphi \sin \theta = \frac{1}{2} \sin \theta \cos \theta =$
 $= \frac{1}{2} \sin \theta \cos \theta - \frac{1}{2} \sin \theta \cos \theta = 0$

2 2 $\frac{1}{2} \cos \theta \cos \varphi \times \cos \varphi \cos \theta + \frac{1}{2} \cos \theta \sin^2 \varphi$
 $+ \frac{1}{2} \sin^2 \theta = (1/2)$

2 3 $\frac{1}{2} \cos \theta \cos \varphi (-\sin \varphi) + \frac{1}{2} \cos \theta \sin \varphi \cos \theta$
 $= 0$

3 1 $\left(-\frac{1}{2}\right) \frac{\sin \varphi}{\sin \theta} \sin \theta \cos \varphi + \left(\frac{1}{2}\right) \left(\frac{\cos \varphi}{\sin \theta}\right)$
 $\times \sin \varphi \times \sin \theta = 0$

3 2 $\left(-\frac{1}{2}\right) \frac{\sin \varphi}{\sin \theta} \cos \varphi \cos \theta +$
 $+ \left(\frac{1}{2}\right) \left(\frac{\cos \varphi}{\sin \theta}\right) \sin \varphi \cos \theta =$
 $= \sin \varphi \cos \theta \left[\frac{1}{2} \times \left(\frac{\cos \varphi}{\sin \theta} \right) - \right.$
 $\left. - \left(\frac{\cos \varphi}{\sin \theta} \right) \right] = 0$

3 3 $\left(-\frac{1}{2}\right) \frac{\sin \varphi}{\sin \theta} (-\sin \varphi) + \frac{1}{2} \times \frac{\cos \varphi}{\sin \theta} \times \cos \varphi$
 $= \frac{1}{2 \sin \theta}$

$$\vec{B} = B_x \vec{i} + B_y \vec{j} + B_z \vec{k}$$

$$\vec{B} = B_\alpha \vec{e}_\alpha + B_\beta \vec{e}_\beta + B_\gamma \vec{e}_\gamma$$

$$\vec{B} = B_z \vec{e}_z + B_\theta \vec{e}_\theta + B_\phi \vec{e}_\phi$$

$$B_x = (\vec{B}, \vec{i}) = B_\alpha (\vec{e}_\alpha, \vec{i}) +$$

$$+ B_\beta (\vec{e}_\beta, \vec{i}) + B_\gamma (\vec{e}_\gamma, \vec{i})$$

$$B_y = (\vec{B}, \vec{j}) = B_\alpha (\vec{e}_\alpha, \vec{j}) +$$

$$+ B_\beta (\vec{e}_\beta, \vec{j}) + B_\gamma (\vec{e}_\gamma, \vec{j})$$

$$B_z = (\vec{B}, \vec{k}) = B_\alpha (\vec{e}_\alpha, \vec{k}) +$$

$$+ B_\beta (\vec{e}_\beta, \vec{k}) + B_\gamma (\vec{e}_\gamma, \vec{k})$$

$$\vec{r} = x \vec{i} + y \vec{j} + z \vec{k}$$

$$\vec{e}_\alpha = \frac{\partial \vec{r} / \partial \alpha}{|\partial \vec{r} / \partial \alpha|}$$

$$(\vec{e}_y, \vec{i}) = \frac{x}{\sqrt{x^2 + y^2 + z^2}}$$

$$(\vec{e}_y, \vec{j}) = \frac{y}{\sqrt{x^2 + y^2 + z^2}}$$

$$\text{div } \vec{B} = \frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} + \frac{\partial B_z}{\partial z}$$

$$(\vec{e}_j, \vec{k}) = \frac{z_j}{\sqrt{x_j^2 + y_j^2 + z_j^2}}$$

(zge $j = \alpha, \beta, \gamma$)

$$\begin{pmatrix} \partial B_i / \partial \alpha \\ \partial B_i / \partial \beta \\ \partial B_i / \partial \gamma \end{pmatrix} = \begin{pmatrix} x_\alpha & y_\alpha & z_\alpha \\ x_\beta & y_\beta & z_\beta \\ x_\gamma & y_\gamma & z_\gamma \end{pmatrix} \begin{pmatrix} \partial B_i / \partial x \\ \partial B_i / \partial y \\ \partial B_i / \partial z \end{pmatrix}$$

$$= A$$

$$\begin{pmatrix} \partial B_i / \partial x \\ \partial B_i / \partial y \\ \partial B_i / \partial z \end{pmatrix} = \begin{pmatrix} x_\alpha & y_\alpha & z_\alpha \\ x_\beta & y_\beta & z_\beta \\ x_\gamma & y_\gamma & z_\gamma \end{pmatrix}^{-1} \times \begin{pmatrix} \partial B_i / \partial \alpha \\ \partial B_i / \partial \beta \\ \partial B_i / \partial \gamma \end{pmatrix}$$

$$= A^{-1}$$

Пример: сферические координаты

$$\begin{cases} x = r \sin \theta \cos \varphi \\ y = r \sin \theta \sin \varphi \\ z = r \cos \theta \end{cases}$$

$$\begin{cases} x_r = \sin \theta \cos \varphi \\ y_r = \sin \theta \sin \varphi \\ z_r = \cos \theta \end{cases}$$

$$\begin{cases} x_\theta = r \cos \theta \cos \varphi \\ y_\theta = r \cos \theta \sin \varphi \\ z_\theta = -r \sin \theta \end{cases}$$

$$\begin{cases} x_\varphi = -r \sin \theta \sin \varphi \\ y_\varphi = r \sin \theta \cos \varphi \\ z_\varphi = 0 \end{cases}$$

$$\sqrt{x_r^2 + y_r^2 + z_r^2} = 1$$

$$\sqrt{x_\theta^2 + y_\theta^2 + z_\theta^2} = r$$

$$\sqrt{x_\varphi^2 + y_\varphi^2 + z_\varphi^2} = r \sin \theta$$

$$B_x = B_r(\vec{e}_r, \vec{L}) + B_\theta(\vec{e}_\theta, \vec{L}) + B_\varphi(\vec{e}_\varphi, \vec{L})$$

$$B_y = B_z (\vec{e}_z, \vec{j}) + B_\theta (\vec{e}_\theta, \vec{j}) + B_\varphi (\vec{e}_\varphi, \vec{j})$$

$$B_z = B_z (\vec{e}_z, \vec{k}) + B_\theta (\vec{e}_\theta, \vec{k}) + B_\varphi (\vec{e}_\varphi, \vec{k})$$

$$(\vec{e}_z, \vec{i}) = \frac{x_z}{\sqrt{x_z^2 + y_z^2 + z_z^2}} = \sin \theta \cos \varphi$$

$$(\vec{e}_\theta, \vec{i}) = \frac{x_\theta}{\sqrt{x_\theta^2 + y_\theta^2 + z_\theta^2}} = \frac{z \cos \theta \cos \varphi}{z} = \cos \theta \cos \varphi$$

$$(\vec{e}_\varphi, \vec{i}) = \frac{x_\varphi}{\sqrt{x_\varphi^2 + y_\varphi^2 + z_\varphi^2}} = -\sin \varphi$$

$$B_x = B_z \sin \theta \cos \varphi + B_\theta \cos \theta \cos \varphi + B_\varphi (-\sin \varphi)$$

$$(\vec{e}_z, \vec{j}) = \frac{yz}{\sqrt{x_z^2 + y_z^2 + z_z^2}} =$$

$$= \sin\theta \sin\varphi$$

$$(\vec{e}_\theta, \vec{j}) = \frac{y_\theta}{\sqrt{x_\theta^2 + y_\theta^2 + z_\theta^2}} =$$

$$= \frac{z \cos\theta \sin\varphi}{z} = \cos\theta \sin\varphi$$

$$(\vec{e}_\varphi, \vec{j}) = \frac{y_\varphi}{\sqrt{x_\varphi^2 + y_\varphi^2 + z_\varphi^2}} =$$

$$= \frac{z \sin\theta \cos\varphi}{z \sin\theta} = \cos\varphi$$

$$B_y = B_z \sin\theta \sin\varphi + B_\theta \cos\theta \sin\varphi +$$

$$+ B_\varphi \cos\varphi$$

$$(\vec{e}_z, \vec{k}) = \frac{z_z}{\sqrt{x_z^2 + y_z^2 + z_z^2}} =$$

$$= \cos\theta$$

$$(\vec{e}_\theta, \vec{k}) = \frac{z_\theta}{\sqrt{x_\theta^2 + y_\theta^2 + z_\theta^2}} =$$

$$= \frac{-z \sin \theta}{z} = -\sin \theta$$

$$(\vec{e}_\varphi, \vec{k}) = \frac{z_\varphi}{\sqrt{x_\varphi^2 + y_\varphi^2 + z_\varphi^2}} = 0$$

$$B_z = B_r \cos \theta - B_\theta \sin \theta$$

$$\partial B_x / \partial z = \frac{\partial B_r}{\partial z} \sin \theta \cos \varphi +$$

$$+ \frac{\partial B_\theta}{\partial z} \cos \theta \cos \varphi - \frac{\partial B_\varphi}{\partial z} \sin \varphi$$

$$\partial B_x / \partial \theta = B_r \cos \theta \cos \varphi +$$

$$+ \frac{\partial B_r}{\partial \theta} \sin \theta \cos \varphi + \frac{\partial B_\theta}{\partial \theta} \cos \theta \cos \varphi +$$

$$+ B_\theta (-\sin \theta) \cos \varphi + \frac{\partial B_\varphi}{\partial \theta} (-\sin \varphi)$$

$$\partial B_x / \partial \varphi = \frac{\partial B_r}{\partial \varphi} \sin \theta \cos \varphi +$$

$$+ B_r \sin \theta (-\sin \varphi) + \frac{\partial B_\theta}{\partial \varphi} \cos \theta \cos \varphi +$$

$$+ B_\theta \cos \theta (-\sin \varphi) -$$

$$- \frac{\partial B_\varphi}{\partial \varphi} \sin \varphi - B_\varphi \cos \varphi$$

$$\frac{\partial B_y}{\partial z} = \frac{\partial B_z}{\partial z} \sin \theta \sin \phi + \frac{\partial B_\theta}{\partial z} \times$$

$$\times \cos \theta \sin \phi + \frac{\partial B_\phi}{\partial z} \cos \phi$$

$$\frac{\partial B_y}{\partial \theta} = \frac{\partial B_z}{\partial \theta} \sin \theta \sin \phi +$$

$$+ B_z \cos \theta \sin \phi + \frac{\partial B_\theta}{\partial \theta} \cos \theta \sin \phi +$$

$$+ B_\theta (-\sin \theta) \sin \phi +$$

$$+ \frac{\partial B_\phi}{\partial \theta} \cos \phi$$

$$\frac{\partial B_y}{\partial \phi} = \frac{\partial B_z}{\partial \phi} \sin \theta \sin \phi +$$

$$+ B_z \sin \theta \cos \phi + \frac{\partial B_\theta}{\partial \phi} \times$$

$$\times \cos \theta \sin \phi + B_\theta \cos \theta \cos \phi +$$

$$+ \frac{\partial B_\phi}{\partial \phi} \cos \phi - B_\phi \sin \phi$$

$$\frac{\partial B_z}{\partial z} = \frac{\partial B_z}{\partial z} \cos \theta - \frac{\partial B_\theta}{\partial z} \sin \theta$$

$$\frac{\partial B_z}{\partial \theta} = \frac{\partial B_z}{\partial \theta} \cos \theta - B_z \sin \theta -$$

$$- \frac{\partial B_\theta}{\partial \theta} \sin \theta - B_\theta \cos \theta$$

$$\frac{\partial B_z}{\partial \phi} = \frac{\partial B_z}{\partial \phi} \cos \theta - \frac{\partial B_\theta}{\partial \phi} \sin \theta$$

$$\begin{pmatrix} \partial B_z / \partial x \\ \partial B_z / \partial y \\ \partial B_z / \partial z \end{pmatrix} = \begin{pmatrix} \sin \theta \cos \phi & \frac{1}{r} \cos \theta \cos \phi & -\frac{1}{r} \sin \theta \\ \sin \theta \sin \phi & \frac{1}{r} \cos \theta \sin \phi & \frac{1}{r} \cos \theta \\ \cos \theta & -\frac{1}{r} \sin \theta & 0 \end{pmatrix}$$

$$= A^{-1}$$

$$\times \begin{pmatrix} \partial B_z / \partial z \\ \partial B_z / \partial \theta \\ \partial B_z / \partial \phi \end{pmatrix}$$

$$\begin{aligned}
\text{div } \vec{B} &= \frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} + \frac{\partial B_z}{\partial z} = \\
&= \left[\sin \theta \cos \varphi \times \left(\frac{\partial B_z}{\partial z} \sin \theta \cos \varphi + \right. \right. \\
&\quad \left. \left. + \frac{\partial B_\theta}{\partial z} \cos \theta \cos \varphi - \frac{\partial B_\varphi}{\partial z} \sin \varphi \right) + \right. \\
&\quad \left. + \frac{1}{r} \cos \theta \cos \varphi \times \right. \\
&\quad \left. \times \left(B_z \cos \theta \cos \varphi + \frac{\partial B_z}{\partial \theta} \times \sin \theta \cos \varphi + \right. \right. \\
&\quad \left. \left. + \frac{\partial B_\theta}{\partial \theta} \cos \theta \cos \varphi - \sin \theta \cos \varphi B_\theta - \right. \right. \\
&\quad \left. \left. - \sin \varphi \frac{\partial B_\varphi}{\partial \theta} \right) - \frac{1}{z} \frac{\sin \varphi}{\sin \theta} \times \right. \\
&\quad \left. \times \left(\frac{\partial B_z}{\partial \varphi} \sin \theta \cos \varphi + B_z (-\sin \varphi) \times \right. \right. \\
&\quad \left. \left. \times \sin \theta + \frac{\partial B_\theta}{\partial \varphi} \cos \theta \cos \varphi - \right. \right. \\
&\quad \left. \left. - \sin \varphi \cos \theta B_\theta - \right. \right. \\
&\quad \left. \left. - \frac{\partial B_\varphi}{\partial \varphi} \sin \varphi - B_\varphi \cos \varphi \right) \right] +
\end{aligned}$$

$$\begin{aligned}
& + \left[\sin \theta \sin \varphi \cdot \left(\frac{\partial B_z}{\partial z} \sin \theta \sin \varphi + \right. \right. \\
& \quad \left. \left. + \frac{\partial B_\theta}{\partial z} \cos \theta \sin \varphi + \frac{\partial B_\varphi}{\partial z} \cos \varphi \right) + \right. \\
& \quad \left. + \frac{1}{z} \cos \theta \sin \varphi \cdot \left(\frac{\partial B_z}{\partial \theta} \sin \theta \sin \varphi + \right. \right. \\
& \quad \left. \left. + B_z \cos \theta \sin \varphi + \frac{\partial B_\theta}{\partial \theta} \cos \theta \sin \varphi - \right. \right. \\
& \quad \left. \left. - \sin \theta \sin \varphi B_\theta + \cos \varphi \frac{\partial B_\varphi}{\partial \theta} \right) + \right. \\
& \quad \left. + \frac{1}{z} \frac{\cos \varphi}{\sin \theta} \cdot \left(\frac{\partial B_z}{\partial \varphi} \sin \theta \sin \varphi + \right. \right. \\
& \quad \left. \left. + B_z \sin \theta \cos \varphi + \frac{\partial B_\theta}{\partial \varphi} \cos \theta \sin \varphi + \right. \right. \\
& \quad \left. \left. + B_\theta \cos \theta \cos \varphi + \frac{\partial B_\varphi}{\partial \varphi} \cos \varphi - \right. \right. \\
& \quad \left. \left. - \sin \varphi B_\varphi \right) \right] + \left[\cos \theta \cdot \left(\frac{\partial B_z}{\partial z} \cos \theta - \right. \right. \\
& \quad \left. \left. - \frac{\partial B_\theta}{\partial z} \sin \theta \right) - \frac{1}{z} \sin \theta \cdot \left(\frac{\partial B_z}{\partial \theta} \cos \theta - \right. \right. \\
& \quad \left. \left. - B_z \sin \theta - \frac{\partial B_\theta}{\partial \theta} \sin \theta - B_\theta \cos \theta \right) \right] =
\end{aligned}$$

$$\begin{aligned}
&= \left\{ \frac{\partial B_z}{\partial z} \times (\sin^2 \theta \cos^2 \phi + \sin^2 \theta \sin^2 \phi + \right. \\
&\quad \left. + \cos^2 \theta) \right\} + \left\{ \frac{\partial B_\theta}{\partial z} \times (\sin \theta \cos \theta \cos^2 \phi + \right. \\
&\quad \left. + \sin \theta \cos \theta \sin^2 \phi - \sin \theta \cos \theta) \right\} + \\
&\quad + \left\{ \frac{\partial B_\phi}{\partial z} \times (-\sin \theta \cos \theta \sin \phi + \right. \\
&\quad \left. + \sin \theta \sin \phi \cos \phi) \right\} + \left\{ \frac{\partial B_z}{\partial \theta} \times \right. \\
&\quad \left. \times \left(\frac{1}{2} \sin \theta \cos \theta \cos^2 \phi + \frac{1}{2} \sin \theta \cos \theta \sin^2 \phi \right. \right. \\
&\quad \left. \left. - \frac{1}{2} \sin \theta \cos \theta \right) \right\} + \left\{ \frac{\partial B_\theta}{\partial \theta} \times \right. \\
&\quad \left. \times \left(\frac{1}{2} \cos^2 \theta \cos^2 \phi + \frac{1}{2} \cos^2 \theta \sin^2 \phi + \right. \right. \\
&\quad \left. \left. + \frac{1}{2} \sin^2 \theta \right) \right\} + \left\{ \frac{\partial B_\phi}{\partial \theta} \times \right. \\
&\quad \left. \times \left(-\frac{1}{2} \cos \theta \cos \phi \sin \phi + \frac{1}{2} \cos \theta \sin \phi \cos \phi \right) \right\} + \\
&\quad + \left\{ \frac{\partial B_z}{\partial \phi} \times \left(-\frac{1}{2} \sin \phi \cos \phi + \frac{1}{2} \cos \phi \sin \phi \right) \right\} + \\
&\quad + \left\{ \frac{\partial B_\theta}{\partial \phi} \times \left(-\frac{1}{2} \times \frac{\sin \phi}{\sin \theta} \cos \theta \cos \phi + \right. \right. \\
&\quad \left. \left. + \frac{1}{2} \cos \phi \cos \theta \times \frac{\sin \phi}{\sin \theta} \right) \right\} +
\end{aligned}$$

$$\begin{pmatrix} f_x \\ f_y \\ f_z \end{pmatrix} = \underbrace{\begin{pmatrix} x_\alpha & y_\alpha & z_\alpha \\ x_\beta & y_\beta & z_\beta \\ x_\gamma & y_\gamma & z_\gamma \end{pmatrix}}_{=A} \begin{pmatrix} f_\alpha \\ f_\beta \\ f_\gamma \end{pmatrix}$$

$$\begin{pmatrix} f_x \\ f_y \\ f_z \end{pmatrix} = \underbrace{\begin{pmatrix} x_\alpha & y_\alpha & z_\alpha \\ x_\beta & y_\beta & z_\beta \\ x_\gamma & y_\gamma & z_\gamma \end{pmatrix}}_{=A^{-1}}^{-1} \begin{pmatrix} f_\alpha \\ f_\beta \\ f_\gamma \end{pmatrix}$$

$$\text{T.K.} \begin{cases} \frac{\partial f}{\partial x} = \frac{\partial f}{\partial \alpha} \cdot \frac{\partial \alpha}{\partial x} + \frac{\partial f}{\partial \beta} \cdot \frac{\partial \beta}{\partial x} + \frac{\partial f}{\partial \gamma} \cdot \frac{\partial \gamma}{\partial x} \\ \frac{\partial f}{\partial y} = \frac{\partial f}{\partial \alpha} \cdot \frac{\partial \alpha}{\partial y} + \frac{\partial f}{\partial \beta} \cdot \frac{\partial \beta}{\partial y} + \frac{\partial f}{\partial \gamma} \cdot \frac{\partial \gamma}{\partial y} \\ \frac{\partial f}{\partial z} = \frac{\partial f}{\partial \alpha} \cdot \frac{\partial \alpha}{\partial z} + \frac{\partial f}{\partial \beta} \cdot \frac{\partial \beta}{\partial z} + \frac{\partial f}{\partial \gamma} \cdot \frac{\partial \gamma}{\partial z} \end{cases}$$

$$\text{mo } A^{-1} = \begin{pmatrix} x_\alpha & y_\alpha & z_\alpha \\ x_\beta & y_\beta & z_\beta \\ x_\gamma & y_\gamma & z_\gamma \end{pmatrix}^{-1} = \begin{pmatrix} \alpha_x & \beta_x & \gamma_x \\ \alpha_y & \beta_y & \gamma_y \\ \alpha_z & \beta_z & \gamma_z \end{pmatrix} \begin{pmatrix} f_\alpha \\ f_\beta \\ f_\gamma \end{pmatrix}$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x} \left(\frac{\partial f}{\partial x} \right) = \frac{\partial}{\partial x} \left(\frac{\partial f}{\partial x} \right) \cdot \frac{\partial x}{\partial x} + \frac{\partial}{\partial \beta} \left(\frac{\partial f}{\partial x} \right) \cdot \frac{\partial \beta}{\partial x} + \frac{\partial}{\partial \gamma} \left(\frac{\partial f}{\partial x} \right) \cdot \frac{\partial \gamma}{\partial x}$$

Т.е. можно записать:

$$\begin{pmatrix} f_{xx} & f_{xy} & f_{xz} \\ f_{yx} & f_{yy} & f_{yz} \\ f_{zx} & f_{zy} & f_{zz} \end{pmatrix} = \begin{pmatrix} \frac{\partial f_x}{\partial x} & \frac{\partial f_x}{\partial \beta} & \frac{\partial f_x}{\partial \gamma} \\ \frac{\partial f_y}{\partial x} & \frac{\partial f_y}{\partial \beta} & \frac{\partial f_y}{\partial \gamma} \\ \frac{\partial f_z}{\partial x} & \frac{\partial f_z}{\partial \beta} & \frac{\partial f_z}{\partial \gamma} \end{pmatrix} \begin{pmatrix} dx & d\beta & d\gamma \\ \beta_x & \beta_y & \beta_z \\ \gamma_x & \gamma_y & \gamma_z \end{pmatrix} = (A^{-1})^T$$

Нам интересуют аргументы элементов, стоящих на диагонали.

Пример: сферические координаты.

$$A^{-1} = \begin{pmatrix} \sin \theta \cos \varphi & \frac{1}{r} \cos \theta \cos \varphi & (-\frac{1}{r}) \cdot \frac{\sin \varphi}{\sin \theta} \\ \sin \theta \sin \varphi & \frac{1}{r} \cos \theta \sin \varphi & \frac{1}{r} \cdot \frac{\cos \varphi}{\sin \theta} \\ \cos \theta & (-\frac{1}{r}) \sin \theta & 0 \end{pmatrix}$$

$$x=r; \beta=\theta; \gamma=\varphi$$

$$\begin{cases} f_x = \sin \theta \cos \varphi f_r + \frac{1}{r} \cos \theta \cos \varphi f_\theta - \frac{1}{r} \cdot \frac{\sin \varphi}{\sin \theta} f_\varphi \\ f_y = \sin \theta \sin \varphi f_r + \frac{1}{r} \cos \theta \sin \varphi f_\theta + \frac{1}{r} \cdot \frac{\cos \varphi}{\sin \theta} f_\varphi \\ f_z = \cos \theta f_r - \frac{1}{r} \sin \theta f_\theta \end{cases}$$

$$\frac{\partial}{\partial r} (f_x) = \sin \theta \cos \varphi f_{rr} + \frac{1}{r} \cos \theta \cos \varphi f_{r\theta} - \frac{1}{r^2} \cos \theta \cos \varphi f_\theta -$$

$$- \frac{1}{r^2} \cdot \frac{\sin \varphi}{\sin \theta} f_{\varphi r} + \frac{1}{r^2} \cdot \frac{\sin \varphi}{\sin \theta} f_\varphi$$

$$\frac{\partial}{\partial r} (f_y) = \sin \theta \sin \varphi f_{rr} + \frac{1}{r} \cos \theta \sin \varphi f_{r\theta} - \frac{1}{r^2} \cos \theta \sin \varphi f_\theta +$$

$$+ \frac{1}{r^2} \cdot \left(\frac{\cos \varphi}{\sin \theta} \right) f_{\varphi r} - \frac{1}{r^2} \cdot \left(\frac{\cos \varphi}{\sin \theta} \right) f_\varphi$$

$$\frac{\partial}{\partial r} (f_z) = \cos \theta f_{rr} - \frac{1}{r} \sin \theta f_{r\theta} + \frac{1}{r^2} \sin \theta f_\theta$$

$$\frac{\partial f_x}{\partial \theta} = \sin \theta \cos \varphi f_{zB} + \cos \theta \cos \varphi f_{zr} - \frac{1}{2} \sin \theta \cos \varphi f_{\theta} + \frac{1}{2} \cos \theta \cos \varphi f_{\theta\theta} - \frac{1}{2} \times \frac{\sin \varphi}{\sin \theta} f_{\varphi\theta} + \frac{1}{2} \times \frac{\sin \varphi}{\sin^2 \theta} f_{\varphi\varphi} \cos \theta$$

$$\frac{\partial f_y}{\partial \theta} = \sin \theta \sin \varphi f_{zB} + \cos \theta \sin \varphi f_{zr} - \frac{1}{2} \sin \theta \sin \varphi f_{\theta} + \frac{1}{2} \cos \theta \sin \varphi f_{\theta\theta} - \frac{1}{2} \times \frac{\cos \varphi}{\sin^2 \theta} \cos \theta f_{\varphi} + \frac{1}{2} \times \frac{\cos \varphi}{\sin \theta} f_{\varphi\theta}$$

$$\frac{\partial f_z}{\partial \theta} = -\sin \theta f_{zr} + \cos \theta f_{z\theta} - \frac{1}{2} \cos \theta f_{\theta} - \frac{1}{2} \sin \theta f_{\theta\theta}$$

$$\frac{\partial f_x}{\partial \varphi} = \sin \theta \cos \varphi f_{z\varphi} + (-\sin \theta \sin \varphi f_{zr}) + \frac{1}{2} \cos \theta (-\sin \varphi) f_{\theta} + \frac{1}{2} \cos \theta \cos \varphi f_{\theta\varphi} - \frac{1}{2} \times \frac{\cos \varphi}{\sin \theta} f_{\varphi} - \frac{1}{2} \frac{\sin \varphi}{\sin \theta} f_{\varphi\varphi}$$

$$\frac{\partial f_y}{\partial \varphi} = \sin \theta \cos \varphi f_{zr} + \sin \theta \sin \varphi f_{z\varphi} + \frac{1}{2} \cos \theta \cos \varphi f_{\theta} + \frac{1}{2} \cos \theta \sin \varphi f_{\theta\varphi} + \frac{1}{2} \times \frac{\cos \varphi}{\sin \theta} f_{\varphi\varphi} - \frac{1}{2} \times \frac{\sin \varphi}{\sin \theta} f_{\varphi}$$

$$\frac{\partial f_z}{\partial \varphi} = \cos \theta f_{z\varphi} - \frac{1}{2} \sin \theta f_{\theta\varphi}$$

$$\begin{cases} f_{xx} = \frac{\partial f_x}{\partial x} \cdot x + \frac{\partial f_x}{\partial \theta} \cdot \theta + \frac{\partial f_x}{\partial \varphi} \cdot \varphi \\ f_{yy} = \frac{\partial f_y}{\partial x} \cdot x + \frac{\partial f_y}{\partial \theta} \cdot \theta + \frac{\partial f_y}{\partial \varphi} \cdot \varphi \\ f_{zz} = \frac{\partial f_z}{\partial x} \cdot x + \frac{\partial f_z}{\partial \theta} \cdot \theta + \frac{\partial f_z}{\partial \varphi} \cdot \varphi \end{cases}$$

$$\begin{pmatrix} x & y & z \\ \theta & \varphi & \end{pmatrix} = \begin{pmatrix} \sin \theta \cos \varphi & \sin \theta \sin \varphi & \cos \theta \\ \frac{1}{2} \cos \theta \cos \varphi & \frac{1}{2} \cos \theta \sin \varphi & (-\frac{1}{2}) \sin \theta \\ (-\frac{1}{2}) \frac{\sin \varphi}{\sin \theta} & \frac{1}{2} \frac{\cos \varphi}{\sin \theta} & 0 \end{pmatrix}$$

$$\begin{aligned}
 f_{xx} &= \sin\theta \cos\varphi \cdot \left(\sin\theta \cos\varphi f_{xx} + \frac{1}{z} \cos\theta \cos\varphi f_{xz} - \right. \\
 &= \frac{1}{z^2} \cos\theta \cos\varphi f_{\theta\theta} - \frac{1}{z} \cdot \frac{\sin\varphi}{\sin\theta} f_{\varphi x} + \frac{1}{z^2} \cdot \frac{\sin\varphi}{\sin\theta} f_{\varphi\theta} \left. + \right. \\
 &+ \frac{1}{z} \cos\theta \cos\varphi \cdot (\sin\theta \cos\varphi f_{z\theta} + \cos\theta \cos\varphi f_{z\varphi} + \\
 &+ \frac{1}{z} \cos\theta \cos\varphi f_{\theta\theta} - \frac{1}{z} \sin\theta \cos\varphi f_{\theta\varphi} - \frac{1}{z} \cdot \frac{\sin\varphi}{\sin\theta} f_{\varphi\theta} + \\
 &+ \frac{1}{z} \cdot \frac{\sin\varphi}{\sin\theta} f_{\varphi\varphi} \cdot \cos\theta) + \left(-\frac{1}{z} \cdot \frac{\sin\varphi}{\sin\theta} \right) \cdot (\sin\theta \cos\varphi f_{x\varphi} - \\
 &= \sin\theta \sin\varphi f_{xz} - \frac{1}{z} \cos\theta \sin\varphi f_{\theta\varphi} + \frac{1}{z} \cos\theta \cos\varphi f_{\theta\varphi} - \\
 &= \frac{1}{z} \cdot \frac{\cos\varphi}{\sin\theta} f_{\varphi\theta} - \frac{1}{z} \cdot \frac{\sin\varphi}{\sin\theta} f_{\varphi\varphi} \left. \right)
 \end{aligned}$$

$$\begin{aligned}
 f_{yy} &= \sin\theta \sin\varphi \cdot \left(\sin\theta \sin\varphi f_{yy} + \frac{1}{z} \cos\theta \sin\varphi f_{yz} - \right. \\
 &= \frac{1}{z^2} \cos\theta \sin\varphi f_{\theta\theta} + \frac{1}{z} \left(\frac{\cos\varphi}{\sin\theta} \right) f_{\varphi z} - \frac{1}{z^2} \left(\frac{\cos\varphi}{\sin\theta} \right) f_{\varphi\theta} \left. + \right. \\
 &+ \frac{1}{z} \cos\theta \sin\varphi \cdot (\sin\theta \sin\varphi f_{z\theta} + \cos\theta \sin\varphi f_{z\varphi} - \\
 &= \frac{1}{z} \sin\theta \sin\varphi f_{\theta\varphi} + \frac{1}{z} \cos\theta \sin\varphi f_{\theta\theta} - \frac{1}{z} \cdot \frac{\cos\varphi}{\sin\theta} \cdot \\
 &\cdot \cos\theta f_{\varphi\theta} + \frac{1}{z} \cdot \frac{\cos\varphi}{\sin\theta} f_{\varphi\varphi} \left. \right) + \frac{1}{z} \cdot \frac{\cos\varphi}{\sin\theta} \cdot \\
 &\cdot \left(\sin\theta \cos\varphi f_{xz} + \sin\theta \sin\varphi f_{x\varphi} + \frac{1}{z} \cos\theta \cos\varphi \cdot \right. \\
 &\cdot f_{\theta\varphi} + \frac{1}{z} \cos\theta \sin\varphi f_{\theta\theta} + \frac{1}{z} \cdot \frac{\cos\varphi}{\sin\theta} f_{\varphi\varphi} - \frac{1}{z} \cdot \frac{\sin\varphi}{\sin\theta} f_{\varphi\theta} \left. \right)
 \end{aligned}$$

$$\begin{aligned}
 f_{zz} &= \cos\theta \cdot \left(\cos\theta f_{zz} - \frac{1}{z} \sin\theta f_{\theta z} + \frac{1}{z^2} \sin\theta f_{\theta\theta} \right) - \\
 &= \frac{1}{z} \sin\theta (-\sin\theta f_{xz} + \cos\theta f_{z\theta}) - \frac{1}{z} \cos\theta f_{\theta\varphi} - \\
 &= \frac{1}{z} \sin\theta f_{\theta\theta}
 \end{aligned}$$

$$\begin{aligned}
\Delta &= f_{xx} + f_{yy} + f_{zz} = f_{zz} (\sin^2 \theta \cos^2 \varphi + \sin^2 \theta \sin^2 \varphi + \cos^2 \theta) + \\
&+ f_{\theta z} \left(\frac{1}{z} \sin \theta \cos \theta \cos^2 \varphi + \frac{1}{z} \sin \theta \cos \theta \sin^2 \varphi + \right. \\
&+ \frac{1}{z} \sin \theta \cos \theta + \sin^2 \varphi + \frac{1}{z} \sin \theta \cos \theta \sin^2 \varphi - \\
&- \frac{1}{z} \sin \theta \cos \theta - \frac{1}{z} \sin \theta \cos \theta \left. \right) + f_{\theta} \left(-\frac{1}{z^2} \cos \theta \sin \theta - \cos^2 \varphi \right. \\
&+ \frac{1}{z^2} \cos \theta \cos^2 \varphi \cdot \left(-\frac{1}{z} \sin \theta \right) + \frac{1}{z^2} \left(\frac{\sin^2 \varphi}{\sin^4 \theta} \cos \theta - \right. \\
&- \frac{1}{z^2} \sin \theta \cos \theta \sin^2 \varphi - \frac{1}{z^2} \sin \theta \cos \theta \sin^2 \varphi \left. f_{\theta} + \right. \\
&+ \frac{1}{z^2} \times \frac{\cos \theta \cos^2 \varphi}{\sin \theta} + \frac{1}{z^2} \sin \theta \cos \theta + \frac{1}{z^2} \sin \theta \cos \theta \left. \right) + \\
&+ f_{\varphi z} \left(\left(-\frac{1}{z} \right) \sin \varphi \cos \varphi - \frac{1}{z} \sin \varphi \cos \varphi + \right. \\
&+ \frac{1}{z} \cos \varphi \sin \varphi + \frac{1}{z} \cos \varphi \sin \varphi \left. \right) + \\
&+ f_{\varphi} \left(\frac{1}{z^2} \sin \varphi \cos \varphi + \frac{1}{z^2} \frac{\cos^2 \theta}{\sin^2 \theta} \sin \varphi \cos \varphi + \right. \\
&+ \frac{1}{z^2} \times \frac{\sin \varphi \cos \varphi}{\sin^2 \theta} - \frac{1}{z^2} \times \sin \varphi \cos \varphi - \frac{1}{z^2} \times \frac{\cos^2 \theta}{\sin^2 \theta} \times \\
&\times \sin \varphi \cos \varphi - \frac{1}{z^2} \left(\sin \varphi \cos \varphi / \sin^2 \theta \right) \left. \right) + \\
&+ f_z \left(\cos^2 \theta \cos^2 \varphi \frac{1}{z} + \frac{1}{z} \sin^2 \varphi + \right. \\
&+ \frac{1}{z} \cos^2 \theta \sin^2 \varphi + \frac{1}{z} \cos^2 \varphi + \frac{1}{z} \sin^2 \varphi \left. \right) + \\
&+ f_{\theta\theta} \left(\frac{1}{z^2} \cos^2 \theta \cos^2 \varphi + \frac{1}{z^2} \sin^2 \theta + \frac{1}{z^2} \cos^2 \theta \sin^2 \varphi \right) + \\
&+ f_{\varphi\theta} \left(-\frac{1}{z^2} \times \frac{\sin \varphi \cos \varphi}{\sin \theta} \cos \theta - \frac{1}{z^2} \times \frac{\sin \varphi \cos \varphi}{\sin \theta} \cos \theta + \right. \\
&+ \frac{1}{z^2} \left(\cos \theta \sin \varphi \cos \varphi / \sin \theta \right) + \frac{1}{z^2} \times \frac{\cos \varphi \sin \varphi \cos \theta}{\sin \theta} \left. \right) + \\
&+ f_{\varphi\varphi} \left(\frac{1}{z^2} \times \frac{\sin^2 \varphi}{\sin^2 \theta} + \frac{1}{z^2} \times \frac{\cos^2 \varphi}{\sin^2 \theta} \right) =
\end{aligned}$$

Градиент функции Φ в цилиндрических координатах.

В декартовых координатах $\text{grad} \Phi = \frac{\partial \Phi}{\partial x} \vec{i} + \frac{\partial \Phi}{\partial y} \vec{j} + \frac{\partial \Phi}{\partial z} \vec{k} =$
 $= \left(\frac{\partial \Phi}{\partial x}, \frac{\partial \Phi}{\partial y}, \frac{\partial \Phi}{\partial z} \right) \begin{pmatrix} \vec{i} \\ \vec{j} \\ \vec{k} \end{pmatrix}$

В общем случае базисные вектора криволинейной системы координат (не нормированные на единицу) связаны с базисными векторами декартовой системы координат следующим образом:

$$\begin{pmatrix} \vec{e}_\alpha \\ \vec{e}_\beta \\ \vec{e}_\gamma \end{pmatrix} = \begin{pmatrix} \frac{\partial x}{\partial \alpha} & \frac{\partial y}{\partial \alpha} & \frac{\partial z}{\partial \alpha} \\ \frac{\partial x}{\partial \beta} & \frac{\partial y}{\partial \beta} & \frac{\partial z}{\partial \beta} \\ \frac{\partial x}{\partial \gamma} & \frac{\partial y}{\partial \gamma} & \frac{\partial z}{\partial \gamma} \end{pmatrix} \begin{pmatrix} \vec{i} \\ \vec{j} \\ \vec{k} \end{pmatrix}, \text{ где } \begin{cases} x = x(\alpha, \beta, \gamma) \\ y = y(\alpha, \beta, \gamma) \\ z = z(\alpha, \beta, \gamma) \end{cases}$$

↑
формулы перехода
от декартовых
координат (x, y, z)
к криволинейным
 (α, β, γ)

= F — обозначим
таким образом
матрицу перехода
от одного базиса
к другому

$$\begin{pmatrix} \vec{i} \\ \vec{j} \\ \vec{k} \end{pmatrix} = F^{-1} \begin{pmatrix} \vec{e}_\alpha \\ \vec{e}_\beta \\ \vec{e}_\gamma \end{pmatrix}$$

Производные связаны так:

$$\begin{pmatrix} \frac{\partial \Phi}{\partial \alpha} & \frac{\partial \Phi}{\partial \beta} & \frac{\partial \Phi}{\partial \gamma} \end{pmatrix} = \begin{pmatrix} \frac{\partial \Phi}{\partial x} & \frac{\partial \Phi}{\partial y} & \frac{\partial \Phi}{\partial z} \end{pmatrix} \begin{pmatrix} \frac{\partial x}{\partial \alpha} & \frac{\partial x}{\partial \beta} & \frac{\partial x}{\partial \gamma} \\ \frac{\partial y}{\partial \alpha} & \frac{\partial y}{\partial \beta} & \frac{\partial y}{\partial \gamma} \\ \frac{\partial z}{\partial \alpha} & \frac{\partial z}{\partial \beta} & \frac{\partial z}{\partial \gamma} \end{pmatrix}$$
$$\begin{pmatrix} \frac{\partial \Phi}{\partial x} & \frac{\partial \Phi}{\partial y} & \frac{\partial \Phi}{\partial z} \end{pmatrix} = \begin{pmatrix} \frac{\partial \Phi}{\partial x} & \frac{\partial \Phi}{\partial y} & \frac{\partial \Phi}{\partial z} \end{pmatrix} \cdot \underbrace{(F^T)^{-1}}_{= (F^{-1})^T} = \underbrace{(F^T)^{-1}}_{= (F^{-1})^T} \begin{pmatrix} \frac{\partial \Phi}{\partial x} & \frac{\partial \Phi}{\partial y} & \frac{\partial \Phi}{\partial z} \end{pmatrix}$$

Цилиндрические координаты:

$$\begin{cases} x = \rho \cos \varphi \\ y = \rho \sin \varphi \\ z = z \end{cases} \quad \begin{cases} \vec{e}_\rho = \cos \varphi \vec{i} + \sin \varphi \vec{j} \\ \vec{e}_\varphi = -\rho \sin \varphi \vec{i} + \rho \cos \varphi \vec{j} \\ \vec{e}_z = \vec{k} \end{cases}$$

$$\text{T.e. } F = \begin{pmatrix} \frac{\partial x}{\partial \alpha} & \frac{\partial y}{\partial \alpha} & \frac{\partial z}{\partial \alpha} \\ \frac{\partial x}{\partial \beta} & \frac{\partial y}{\partial \beta} & \frac{\partial z}{\partial \beta} \\ \frac{\partial x}{\partial \gamma} & \frac{\partial y}{\partial \gamma} & \frac{\partial z}{\partial \gamma} \end{pmatrix} = \begin{pmatrix} \cos \varphi & \sin \varphi & 0 \\ -\rho \sin \varphi & \rho \cos \varphi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\det F = \rho \cos^2 \varphi + \rho \sin^2 \varphi = \rho$$

Градиент Φ в криволинейных координатах

$$\text{grad} \Phi = \left(\frac{\partial \Phi}{\partial \alpha} \quad \frac{\partial \Phi}{\partial \beta} \quad \frac{\partial \Phi}{\partial \gamma} \right) \begin{pmatrix} \vec{l} \\ \vec{m} \\ \vec{n} \end{pmatrix} =$$

$$= \left(\frac{\partial \Phi}{\partial \alpha} \quad \frac{\partial \Phi}{\partial \beta} \quad \frac{\partial \Phi}{\partial \gamma} \right) (F^{-1})^T (F^{-1}) \begin{pmatrix} \vec{e}_\alpha \\ \vec{e}_\beta \\ \vec{e}_\gamma \end{pmatrix}, \text{ где}$$

Базисные вектора не нормированные на 1 $\begin{pmatrix} \vec{e}_\alpha \\ \vec{e}_\beta \\ \vec{e}_\gamma \end{pmatrix}$ связаны с нормированными на 1 $\begin{pmatrix} \vec{e}'_\alpha \\ \vec{e}'_\beta \\ \vec{e}'_\gamma \end{pmatrix}$ следующим

образом: $\begin{pmatrix} \vec{e}_\alpha \\ \vec{e}_\beta \\ \vec{e}_\gamma \end{pmatrix} = \begin{pmatrix} \sqrt{g_{\alpha\alpha}} \vec{e}'_\alpha \\ \sqrt{g_{\beta\beta}} \vec{e}'_\beta \\ \sqrt{g_{\gamma\gamma}} \vec{e}'_\gamma \end{pmatrix}$; $g_{\alpha\alpha}, g_{\beta\beta}, g_{\gamma\gamma}$ — компоненты метрического тензора

$$\begin{cases} g_{\alpha\alpha} = (\vec{e}_\alpha, \vec{e}_\alpha) = \left(\frac{\partial x}{\partial \alpha} \right)^2 + \left(\frac{\partial y}{\partial \alpha} \right)^2 + \left(\frac{\partial z}{\partial \alpha} \right)^2 = 1 \\ g_{\beta\beta} = (\vec{e}_\beta, \vec{e}_\beta) = \left(\frac{\partial x}{\partial \beta} \right)^2 + \left(\frac{\partial y}{\partial \beta} \right)^2 + \left(\frac{\partial z}{\partial \beta} \right)^2 = \rho^2 \\ g_{\gamma\gamma} = (\vec{e}_\gamma, \vec{e}_\gamma) = \left(\frac{\partial x}{\partial \gamma} \right)^2 + \left(\frac{\partial y}{\partial \gamma} \right)^2 + \left(\frac{\partial z}{\partial \gamma} \right)^2 = 1 \end{cases}$$

Итак, для цилиндрических координат:

$$F = \begin{pmatrix} \cos\varphi & \sin\varphi & 0 \\ -\rho\sin\varphi & \rho\cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$F^{-1} = \frac{1}{\det F} \begin{pmatrix} (-1)^{1+1} \rho \cos\varphi & (-1)^{1+2} \rho \sin\varphi & (-1)^{1+3} \cdot 0 \\ (-1)^{2+1} \sin\varphi & (-1)^{2+2} \cos\varphi & (-1)^{2+3} \cdot 0 \\ (-1)^{3+1} \cdot 0 & (-1)^{3+2} \cdot 0 & (-1)^{3+3} \cdot \rho \end{pmatrix}^T =$$

$$= \frac{1}{\rho} \begin{pmatrix} \rho \cos\varphi & \rho \sin\varphi & 0 \\ -\sin\varphi & \cos\varphi & 0 \\ 0 & 0 & \rho \end{pmatrix} = \begin{pmatrix} \cos\varphi & -\frac{1}{\rho} \sin\varphi & 0 \\ \sin\varphi & \frac{1}{\rho} \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Проверка: $F F^{-1} = \begin{pmatrix} \cos\varphi & \sin\varphi & 0 \\ -\rho\sin\varphi & \rho\cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\varphi & -\frac{1}{\rho}\sin\varphi & 0 \\ \sin\varphi & \frac{1}{\rho}\cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix} =$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = I \Rightarrow \text{написан } F^{-1} \text{ нами верно.}$$

$$(F^{-1})^T (F^{-1}) = \begin{pmatrix} \cos\varphi & \sin\varphi & 0 \\ -\frac{1}{\rho}\sin\varphi & \frac{1}{\rho}\cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\varphi & -\frac{1}{\rho}\sin\varphi & 0 \\ \sin\varphi & \frac{1}{\rho}\cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix} =$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/\rho^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\text{grad}\varphi = \begin{pmatrix} \frac{\partial\varphi}{\partial\rho} & \frac{\partial\varphi}{\partial\varphi} & \frac{\partial\varphi}{\partial z} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/\rho^2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \vec{e}_\rho \\ \rho \vec{e}_\varphi \\ \vec{e}_z \end{pmatrix} =$$

$$= \begin{pmatrix} \frac{\partial\varphi}{\partial\rho} & \frac{\partial\varphi}{\partial\varphi} & \frac{\partial\varphi}{\partial z} \end{pmatrix} \begin{pmatrix} \vec{e}_\rho \\ (1/\rho) \vec{e}_\varphi \\ \vec{e}_z \end{pmatrix} =$$

$$= \frac{\partial\varphi}{\partial\rho} \cdot \vec{e}_\rho + \frac{1}{\rho} \frac{\partial\varphi}{\partial\varphi} \cdot \vec{e}_\varphi + \frac{\partial\varphi}{\partial z} \cdot \vec{e}_z$$

Ротор вектора $\vec{A}$ в цилиндрических координатах.

Известно, что в декартовых координатах

$$\operatorname{rot} \vec{A} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix} = \vec{i} \cdot \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} \right) +$$

$$+ \vec{j} \cdot \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x} \right) + \vec{k} \cdot \left(-\frac{\partial A_x}{\partial y} + \frac{\partial A_y}{\partial x} \right)$$

Базисные вектора, не нормированные на единицу, криволинейной системы координат (α, β, γ) связаны с декартовыми базисными векторами $\vec{i}, \vec{j}, \vec{k}$ следующим образом:

$$\begin{pmatrix} \vec{e}_\alpha \\ \vec{e}_\beta \\ \vec{e}_\gamma \end{pmatrix} = \begin{pmatrix} \frac{\partial x}{\partial \alpha} & \frac{\partial y}{\partial \alpha} & \frac{\partial z}{\partial \alpha} \\ \frac{\partial x}{\partial \beta} & \frac{\partial y}{\partial \beta} & \frac{\partial z}{\partial \beta} \\ \frac{\partial x}{\partial \gamma} & \frac{\partial y}{\partial \gamma} & \frac{\partial z}{\partial \gamma} \end{pmatrix} \begin{pmatrix} \vec{i} \\ \vec{j} \\ \vec{k} \end{pmatrix} \Rightarrow \begin{pmatrix} \vec{i} \\ \vec{j} \\ \vec{k} \end{pmatrix} = F^{-1} \begin{pmatrix} \vec{e}_\alpha \\ \vec{e}_\beta \\ \vec{e}_\gamma \end{pmatrix}$$

$= F$ — обозначим т.о. матрицу перехода от одного базиса к другому

$$\begin{pmatrix} dx \\ dy \\ dz \end{pmatrix} = \underbrace{\begin{pmatrix} \frac{\partial x}{\partial \alpha} & \frac{\partial x}{\partial \beta} & \frac{\partial x}{\partial \gamma} \\ \frac{\partial y}{\partial \alpha} & \frac{\partial y}{\partial \beta} & \frac{\partial y}{\partial \gamma} \\ \frac{\partial z}{\partial \alpha} & \frac{\partial z}{\partial \beta} & \frac{\partial z}{\partial \gamma} \end{pmatrix}}_{= F^T} \begin{pmatrix} d\alpha \\ d\beta \\ d\gamma \end{pmatrix} \Rightarrow \begin{pmatrix} d\alpha \\ d\beta \\ d\gamma \end{pmatrix} = \underbrace{\left(F^T \right)^{-1}}_{= (F^{-1})^T} \begin{pmatrix} dx \\ dy \\ dz \end{pmatrix}$$



с другой стороны,

$$\begin{pmatrix} d\alpha \\ d\beta \\ d\gamma \end{pmatrix} = \begin{pmatrix} \frac{\partial \alpha}{\partial x} & \frac{\partial \alpha}{\partial y} & \frac{\partial \alpha}{\partial z} \\ \frac{\partial \beta}{\partial x} & \frac{\partial \beta}{\partial y} & \frac{\partial \beta}{\partial z} \\ \frac{\partial \gamma}{\partial x} & \frac{\partial \gamma}{\partial y} & \frac{\partial \gamma}{\partial z} \end{pmatrix} \begin{pmatrix} dx \\ dy \\ dz \end{pmatrix} \Rightarrow \begin{pmatrix} \frac{\partial \alpha}{\partial x} & \frac{\partial \alpha}{\partial y} & \frac{\partial \alpha}{\partial z} \\ \frac{\partial \beta}{\partial x} & \frac{\partial \beta}{\partial y} & \frac{\partial \beta}{\partial z} \\ \frac{\partial \gamma}{\partial x} & \frac{\partial \gamma}{\partial y} & \frac{\partial \gamma}{\partial z} \end{pmatrix} = (F^T)^{-1}$$

$$\vec{A} = A_x \cdot \vec{i} + A_y \cdot \vec{j} + A_z \cdot \vec{k} = A_\alpha \cdot \vec{e}_\alpha + A_\beta \cdot \vec{e}_\beta + A_\gamma \cdot \vec{e}_\gamma \quad / \cdot (\vec{i}, \vec{j}, \vec{k})$$

$\Downarrow$

$$\begin{pmatrix} A_x \\ A_y \\ A_z \end{pmatrix} = \underbrace{\begin{pmatrix} \frac{\partial x}{\partial \alpha} & \frac{\partial x}{\partial \beta} & \frac{\partial x}{\partial \gamma} \\ \frac{\partial y}{\partial \alpha} & \frac{\partial y}{\partial \beta} & \frac{\partial y}{\partial \gamma} \\ \frac{\partial z}{\partial \alpha} & \frac{\partial z}{\partial \beta} & \frac{\partial z}{\partial \gamma} \end{pmatrix}}_{= F^T} \begin{pmatrix} A_\alpha \\ A_\beta \\ A_\gamma \end{pmatrix}$$

Полее, для производных можно написать:

$$\begin{pmatrix} \frac{\partial A_x}{\partial x} & \frac{\partial A_x}{\partial y} & \frac{\partial A_x}{\partial z} \\ \frac{\partial A_y}{\partial x} & \frac{\partial A_y}{\partial y} & \frac{\partial A_y}{\partial z} \\ \frac{\partial A_z}{\partial x} & \frac{\partial A_z}{\partial y} & \frac{\partial A_z}{\partial z} \end{pmatrix} = \begin{pmatrix} \frac{\partial A_x}{\partial \alpha} & \frac{\partial A_x}{\partial \beta} & \frac{\partial A_x}{\partial \gamma} \\ \frac{\partial A_y}{\partial \alpha} & \frac{\partial A_y}{\partial \beta} & \frac{\partial A_y}{\partial \gamma} \\ \frac{\partial A_z}{\partial \alpha} & \frac{\partial A_z}{\partial \beta} & \frac{\partial A_z}{\partial \gamma} \end{pmatrix} \times \underbrace{\begin{pmatrix} \frac{\partial \alpha}{\partial x} & \frac{\partial \alpha}{\partial y} & \frac{\partial \alpha}{\partial z} \\ \frac{\partial \beta}{\partial x} & \frac{\partial \beta}{\partial y} & \frac{\partial \beta}{\partial z} \\ \frac{\partial \gamma}{\partial x} & \frac{\partial \gamma}{\partial y} & \frac{\partial \gamma}{\partial z} \end{pmatrix}}_{=(F^T)^{-1}}$$

Теперь переходим непосредственно к цилиндрическим координатам. Матрица F перехода от одного базиса к другому в этих координатах:

$$F = \begin{pmatrix} \cos \varphi & \sin \varphi & 0 \\ -\rho \sin \varphi & \rho \cos \varphi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Обратная матрица:

$$F^{-1} = \begin{pmatrix} \cos \varphi & -\frac{1}{\rho} \sin \varphi & 0 \\ \sin \varphi & \frac{1}{\rho} \cos \varphi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\begin{cases} A_x = \cos\varphi A_p - p \sin\varphi A_\varphi \\ A_y = \sin\varphi A_p + p \cos\varphi A_\varphi \\ A_z = A_z \end{cases}$$

$$p = \alpha:$$

$$\frac{\partial A_x}{\partial p} = -\sin\varphi A_\varphi + \cos\varphi \frac{\partial A_p}{\partial p} - p \sin\varphi \frac{\partial A_\varphi}{\partial p}$$

$$\frac{\partial A_y}{\partial p} = \cos\varphi A_\varphi + \sin\varphi \frac{\partial A_p}{\partial p} + p \cos\varphi \frac{\partial A_\varphi}{\partial p}$$

$$\frac{\partial A_z}{\partial p} = \frac{\partial A_z}{\partial p}$$

$$\varphi = \beta:$$

$$\frac{\partial A_x}{\partial \varphi} = -\sin\varphi A_p - p \cos\varphi A_\varphi + \cos\varphi \frac{\partial A_p}{\partial \varphi} - p \sin\varphi \frac{\partial A_\varphi}{\partial \varphi}$$

$$\frac{\partial A_y}{\partial \varphi} = \cos\varphi A_p - p \sin\varphi A_\varphi + \sin\varphi \frac{\partial A_p}{\partial \varphi} + p \cos\varphi \frac{\partial A_\varphi}{\partial \varphi}$$

$$\frac{\partial A_z}{\partial \varphi} = \frac{\partial A_z}{\partial \varphi}$$

$$z = \gamma:$$

$$\frac{\partial A_x}{\partial z} = \cos\varphi \frac{\partial A_p}{\partial z} - p \sin\varphi \frac{\partial A_\varphi}{\partial z}$$

$$\frac{\partial A_y}{\partial z} = \sin\varphi \frac{\partial A_p}{\partial z} + p \cos\varphi \frac{\partial A_\varphi}{\partial z}$$

$$\frac{\partial A_z}{\partial z} = \frac{\partial A_z}{\partial z}$$

$$\begin{pmatrix} \frac{\partial A_x}{\partial x} & \frac{\partial A_x}{\partial y} & \frac{\partial A_x}{\partial z} \\ \frac{\partial A_y}{\partial x} & \frac{\partial A_y}{\partial y} & \frac{\partial A_y}{\partial z} \\ \frac{\partial A_z}{\partial x} & \frac{\partial A_z}{\partial y} & \frac{\partial A_z}{\partial z} \end{pmatrix} =$$

$$= \begin{pmatrix} -\sin\varphi A_\varphi + \cos\varphi \frac{\partial A_\varphi}{\partial \rho} - \rho \sin\varphi \frac{\partial A_\varphi}{\partial \rho} & -\sin\varphi A_\rho - \rho \cos\varphi A_\varphi + \cos\varphi \frac{\partial A_\rho}{\partial \varphi} - \rho \sin\varphi \frac{\partial A_\varphi}{\partial \varphi} & \cos\varphi \frac{\partial A_\rho}{\partial z} - \rho \sin\varphi \frac{\partial A_\varphi}{\partial z} \\ \cos\varphi A_\rho + \sin\varphi \frac{\partial A_\rho}{\partial \rho} + \rho \cos\varphi \frac{\partial A_\varphi}{\partial \rho} & \cos\varphi A_\varphi - \rho \sin\varphi A_\rho + \sin\varphi \frac{\partial A_\rho}{\partial \varphi} + \rho \cos\varphi \frac{\partial A_\varphi}{\partial \varphi} & \sin\varphi \frac{\partial A_\rho}{\partial z} + \rho \cos\varphi \frac{\partial A_\varphi}{\partial z} \\ \frac{\partial A_z}{\partial \rho} & \frac{\partial A_z}{\partial \varphi} & \frac{\partial A_z}{\partial z} \end{pmatrix} \times$$

$$\times \begin{pmatrix} \cos\varphi & \sin\varphi & 0 \\ -\frac{1}{\rho} \sin\varphi & \frac{1}{\rho} \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\begin{aligned} \frac{\partial A_x}{\partial y} &= (-\sin\varphi A_\varphi + \cos\varphi \frac{\partial A_\varphi}{\partial \rho} - \rho \sin\varphi \frac{\partial A_\varphi}{\partial \rho}) \times \sin\varphi + \\ &+ (-\sin\varphi A_\rho - \rho \cos\varphi A_\varphi + \cos\varphi \frac{\partial A_\rho}{\partial \varphi} - \rho \sin\varphi \frac{\partial A_\varphi}{\partial \varphi}) \times \frac{1}{\rho} \cos\varphi = \\ &= A_\varphi \times (-\sin^2\varphi - \cos^2\varphi) + A_\rho \times \frac{1}{\rho} (-\sin\varphi) \cos\varphi + \frac{\partial A_\rho}{\partial \rho} \sin\varphi \cos\varphi - \rho \sin^2\varphi \frac{\partial A_\varphi}{\partial \rho} + \\ &+ \frac{1}{\rho} \cos^2\varphi \frac{\partial A_\rho}{\partial \varphi} - \sin\varphi \cos\varphi \frac{\partial A_\varphi}{\partial \varphi} \end{aligned}$$

$$\frac{\partial A_x}{\partial z} = \cos\varphi \frac{\partial A_\rho}{\partial z} - \rho \sin\varphi \frac{\partial A_\varphi}{\partial z}$$

$$\frac{\partial A_y}{\partial z} = \sin\varphi \frac{\partial A_\rho}{\partial z} + \rho \cos\varphi \frac{\partial A_\varphi}{\partial z}$$

$$\begin{aligned}\frac{\partial A_y}{\partial x} &= (\cos\varphi A_\varphi + \sin\varphi \frac{\partial A_\rho}{\partial \rho} + \rho \cos\varphi \frac{\partial A_\varphi}{\partial \rho}) \times \cos\varphi + \\ &+ (\cos\varphi A_\rho - \rho \sin\varphi A_\varphi + \sin\varphi \frac{\partial A_\rho}{\partial \varphi} + \rho \cos\varphi \frac{\partial A_\varphi}{\partial \varphi}) \times (-\frac{1}{\rho} \sin\varphi) = \\ &= A_\varphi (\cos^2\varphi + \sin^2\varphi) + (-\frac{1}{\rho} \sin\varphi) \cos\varphi A_\rho + \sin\varphi \cos\varphi \frac{\partial A_\rho}{\partial \rho} + \rho \cos^2\varphi \frac{\partial A_\varphi}{\partial \rho} - \\ &- \frac{1}{\rho} \sin^2\varphi \frac{\partial A_\rho}{\partial \varphi} - \sin\varphi \cos\varphi \frac{\partial A_\varphi}{\partial \varphi}\end{aligned}$$

$$\frac{\partial A_z}{\partial x} = \frac{\partial A_z}{\partial \rho} \cos\varphi + \frac{\partial A_z}{\partial \varphi} \times (-\frac{1}{\rho} \sin\varphi)$$

$$\frac{\partial A_z}{\partial y} = \frac{\partial A_z}{\partial \rho} \sin\varphi + \frac{\partial A_z}{\partial \varphi} \frac{1}{\rho} \cos\varphi$$

$$\begin{cases} \vec{i} = \cos\varphi \vec{e}_\rho - \frac{1}{\rho} \sin\varphi \vec{e}_\varphi \\ \vec{j} = \sin\varphi \vec{e}_\rho + \frac{1}{\rho} \cos\varphi \vec{e}_\varphi \\ \vec{k} = \vec{e}_z \end{cases}$$

$$\text{Thus, } \text{rot } \vec{A} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix} = \vec{i} \cdot \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} \right) +$$

$$+ \vec{j} \cdot \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x} \right) + \vec{k} \cdot \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \right) =$$

$$= \vec{i} \cdot \left(\frac{\partial A_z}{\partial \rho} \sin\varphi + \frac{\partial A_z}{\partial \varphi} \frac{1}{\rho} \cos\varphi - \sin\varphi \frac{\partial A_\rho}{\partial z} - \rho \cos\varphi \frac{\partial A_\varphi}{\partial z} \right) +$$

$$+ \vec{j} \cdot \left(\cos\varphi \frac{\partial A_\rho}{\partial z} - \rho \sin\varphi \frac{\partial A_\varphi}{\partial z} - \frac{\partial A_z}{\partial \rho} \cos\varphi + \frac{1}{\rho} \sin\varphi \frac{\partial A_z}{\partial \varphi} \right) +$$

$$+ \vec{k} \cdot \left(A_\varphi - \frac{1}{\rho} \sin\varphi \cos\varphi A_\rho + \sin\varphi \cos\varphi \frac{\partial A_\rho}{\partial \rho} + \rho \cos^2\varphi \frac{\partial A_\varphi}{\partial \rho} - \right.$$

$$\left. - \frac{1}{\rho} \sin^2\varphi \frac{\partial A_\rho}{\partial \varphi} - \sin\varphi \cos\varphi \frac{\partial A_\varphi}{\partial \varphi} + A_\rho + A_\rho \frac{1}{\rho} \sin\varphi \cos\varphi - \right.$$

$$\left. - \frac{\partial A_\rho}{\partial \rho} \sin\varphi \cos\varphi + \rho \sin^2\varphi \frac{\partial A_\varphi}{\partial \rho} - \frac{1}{\rho} \cos^2\varphi \frac{\partial A_\rho}{\partial \varphi} + \sin\varphi \cos\varphi \frac{\partial A_\varphi}{\partial \varphi} \right) =$$

$$\begin{aligned}
&= (\cos\varphi \vec{e}_\rho - \frac{1}{\rho} \sin\varphi \vec{e}_\varphi) \times \left(\frac{\partial A_z}{\partial \rho} \sin\varphi + \frac{\partial A_z}{\partial \varphi} \frac{1}{\rho} \cos\varphi - \sin\varphi \frac{\partial A_\rho}{\partial z} - \rho \cos\varphi \frac{\partial A_\varphi}{\partial z} \right) \\
&+ (\sin\varphi \vec{e}_\rho + \frac{1}{\rho} \cos\varphi \vec{e}_\varphi) \times \left(\cos\varphi \frac{\partial A_\rho}{\partial z} - \rho \sin\varphi \frac{\partial A_\varphi}{\partial z} - \frac{\partial A_z}{\partial \rho} \cos\varphi + \frac{1}{\rho} \sin\varphi \frac{\partial A_z}{\partial \varphi} \right) \\
&+ \vec{e}_z \cdot \left(2A_\varphi - \frac{1}{\rho} \frac{\partial A_\rho}{\partial \varphi} + \rho \frac{\partial A_\varphi}{\partial \rho} \right) = \vec{e}_\rho \times \left\{ \frac{\partial A_z}{\partial \rho} \sin\varphi \cos\varphi + \frac{\partial A_z}{\partial \varphi} \frac{1}{\rho} \cos^2\varphi - \right. \\
&- \sin\varphi \cos\varphi \frac{\partial A_\rho}{\partial z} - \rho \cos^2\varphi \frac{\partial A_\varphi}{\partial z} + \sin\varphi \cos\varphi \frac{\partial A_\rho}{\partial z} - \rho \sin^2\varphi \frac{\partial A_\varphi}{\partial z} - \\
&- \frac{\partial A_z}{\partial \rho} \sin\varphi \cos\varphi + \frac{1}{\rho} \sin^2\varphi \frac{\partial A_z}{\partial \varphi} \left. \right\} + \vec{e}_\varphi \times \left\{ \frac{\partial A_z}{\partial \rho} \left(-\frac{1}{\rho} \right) \sin^2\varphi + \right. \\
&+ \frac{\partial A_z}{\partial \varphi} \left(-\frac{1}{\rho^2} \right) \sin\varphi \cos\varphi + \frac{\partial A_\rho}{\partial z} \times \frac{1}{\rho} \sin^2\varphi + \sin\varphi \cos\varphi \frac{\partial A_\varphi}{\partial z} + \\
&+ \frac{1}{\rho} \cos^2\varphi \frac{\partial A_\rho}{\partial z} - \sin\varphi \cos\varphi \frac{\partial A_\varphi}{\partial z} - \frac{1}{\rho} \cos^2\varphi \frac{\partial A_z}{\partial \rho} + \frac{1}{\rho^2} \sin\varphi \cos\varphi \frac{\partial A_z}{\partial \varphi} \left. \right\} + \\
&+ \vec{e}_z \times \left\{ 2A_\varphi - \frac{1}{\rho} \frac{\partial A_\rho}{\partial \varphi} + \rho \frac{\partial A_\varphi}{\partial \rho} \right\} = \vec{e}_\rho \times \left(\frac{1}{\rho} \frac{\partial A_z}{\partial \rho} - \rho \frac{\partial A_\varphi}{\partial z} \right) + \\
&= \vec{e}_\rho \times \left(\frac{1}{\rho} \frac{\partial A_\rho}{\partial z} - \frac{1}{\rho} \frac{\partial A_z}{\partial \rho} \right) + \vec{e}_z \cdot \left(2A_\varphi - \frac{1}{\rho} \frac{\partial A_\rho}{\partial \varphi} + \rho \frac{\partial A_\varphi}{\partial \rho} \right) \\
&= \vec{e}_\rho \times \left(\frac{1}{\rho} \frac{\partial A_\rho}{\partial z} - \frac{1}{\rho} \frac{\partial A_z}{\partial \rho} \right) + \vec{e}_z \cdot \left(2A_\varphi - \frac{1}{\rho} \frac{\partial A_\rho}{\partial \varphi} + \rho \frac{\partial A_\varphi}{\partial \rho} \right)
\end{aligned}$$

В данном случае координаты вектора $\vec{A}$ определены через его разложение $\vec{A} = A_r \vec{e}_r + A_\varphi \vec{e}_\varphi + A_z \vec{e}_z$, где базисные вектора $\vec{e}_r, \vec{e}_\varphi, \vec{e}_z$ не нормированы на единицу.

Иногда координаты определяют таким образом:

$$\vec{A} = A'_y \vec{e}'_y + A'_x \vec{e}'_x + A'_z \vec{e}'_z, \text{ где } \vec{e}'_y, \vec{e}'_x, \vec{e}'_z - \text{нормированы на } 1.$$

Важно в сферич. системе координат такая:

$$A_j = A_j'; \vec{e}_j = \vec{e}_j'; A_\varphi = \frac{A_\varphi'}{S}; \vec{e}_\varphi = S \vec{e}_\varphi'; A_z = A_z'; \vec{e}_z = \vec{e}_z'$$

Тогда выражение для ротора принимает вид:

$$\text{rot } \vec{A} = \frac{1}{\rho} \left(\frac{\partial A_z'}{\partial \varphi} - \frac{\partial A_\varphi'}{\partial z} \right) \vec{e}_\rho + \left(\frac{\partial A_\rho'}{\partial z} - \frac{\partial A_z'}{\partial \rho} \right) \vec{e}_\varphi + \left(\frac{\partial A_\varphi'}{\partial \rho} - \frac{1}{\rho} \frac{\partial A_\rho}{\partial \varphi} + \rho \frac{\partial (A_\varphi'/\rho)}{\partial \rho} \right) \vec{e}_z$$