On solutions of biharmonic problems

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Abstract: For solving biharmonic problems with application to radar imaging, we need to solve boundary value problems for the Poisson equation using the scattering model. In addition, no information about boundary values is available. In order to select suitable solutions, we solve the Poisson equation under the side condition that some criterion function, usually a Sobolev norm, should be minimized. Under appropriate smoothness assumptions these problems may be reformulated as boundary value problems for the biharmonic equation.

1. Introduction

Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, be a bounded Lipschitz domain with connected boundary $\partial\Omega$, and $\Omega \cup \partial\Omega = \overline{\Omega} =$ is the closure of Ω. We consider the following boundary value problems for the biharmonic equation in Lipschitz domains:

$$\Delta^2 u(x) = f(x), \qquad x \in \Omega \tag{1}$$

with the Navier boundary conditions

$$u = g, \quad Mu \equiv \sigma\Delta u + (1-\sigma)\frac{\partial^2 u}{\partial \nu^2} = h_1, \quad \text{on} \quad \partial\Omega, \tag{2}$$

or the Neumann boundary conditions

$$Mu \equiv \sigma\Delta u + (1-\sigma)\frac{\partial^2 u}{\partial \nu^2} = h_1, \quad Nu \equiv \frac{\partial \Delta u}{\partial \nu} + (1-\sigma)\,\mathrm{div}_{\partial\Omega}(\partial^2 u \cdot \nu)_{\partial\Omega} = h_2, \quad \text{on} \quad \partial\Omega, \tag{3}$$

where ν is the outward unit normal to $\partial\Omega$, and $\frac{1}{n-1} < \sigma < 1$, σ is a constant known as the Poisson ratio. A unique solution u (modulo linear functions) is obtained in the class of solutions with non-tangential maximal function of the second-order derivatives in $L^p(\partial\Omega)$. The corresponding Poisson problem is well-posed unless $\sigma = 1$.

Note that standard elliptic regularity results are available in [3]. This monograph covers higher order linear and nonlinear elliptic boundary value problems, mainly with the biharmonic or polyharmonic operator as leading principal part. Underlying models and, in particular, the role of different boundary conditions are explained in detail. As for linear problems, after a brief summary of the existence theory and L^p and Schauder estimates, the focus is on positivity. The required kernel estimates are also presented in detail.

Boundary value problems for a biharmonic (polyharmonic) equation in unbounded domains are studied in [7]– [12], in which the condition of the boundedness of the following weighted Dirichlet integral of solution is finite, namely

$$\int_\Omega |x|^a |\partial u|^2 \, dx < \infty, \quad a \in \mathbb{R},$$

where $a \in \mathbb{R}$ is a fixed number and $|\partial u|^2$ denotes the Frobenius norm of the Hessian matrix of u. The author in [7]– [12] investigates the dimension of the space of the solutions to the boundary value problems for a biharmonic (polyharmonic) equation, providing explicit formulas which depends on n and a.

Elliptic problems with parameters in the boundary conditions are called Steklov problems from their first appearance in [18]. In the case of the biharmonic operator, these conditions were first considered in [1], [6] and [16], who studied the isoperimetric properties of the first eigenvalue.

In [2] the boundary value problems for the biharmonic equation and the Stokes system are studied in a half space, and, using the Schwartz reflection principle in weighted L^q -space the uniqueness of solutions of the Stokes system or the biharmonic equation is proved.

Notation: $C_0^\infty(\Omega)$ is the space of infinitely differentiable functions in Ω with compact support in Ω; $H^m(\Omega)$ is the Sobolev space obtained by the completion of $C^\infty(\overline{\Omega})$ with respect to the norm

$$\|u(x); H^m(\Omega)\| = \left(\int\limits_\Omega \sum_{|\alpha| \le m} |\partial^\alpha u(x)|^2 dx \right)^{1/2}, \; m = 1, 2,$$

where $\partial^\alpha \equiv \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \dots \partial x_n^{\alpha_n}}$, $\alpha = (\alpha_1, \dots, \alpha_n)$ is a multi-index, $\alpha_i \ge 0$ are integers, and $|\alpha| = \alpha_1 + \cdots + \alpha_n$;

$\overset{\circ}{H}{}^m(\Omega)$ is the space obtained by the completion of $C_0^\infty(\Omega)$ with respect to the norm $||u(x); H^m(\Omega)||$;

Definition 1 *A solution of the homogenous biharmonic equation* (1) *in* Ω *is a function* $u \in H^2(\Omega)$ *such that for every function* $\varphi \in C_0^\infty(\Omega)$, *the following integral identity holds:*

$$\int_\Omega \Delta u \, \Delta \varphi \, dx = \int_\Omega f \, \varphi \, dx.$$

In Section 2 we will derive the mathematical model used for describing the radar process. In our parametrization the unknown is the height function H. As will be shown in Section 2 the height function is determined in two steps. In the first step $\mathfrak{L}(H)$, with $\mathfrak{L}$ a certain second-order differential operator, is determined. After retrieving H the equation $\mathfrak{L}(H) = f$ must be solved. To a good approximation the operator $\mathfrak{L}$ can be replaced by the Laplacian. So the second step simply consists of solving the Poisson equation over some smooth bounded domain, usually a rectangular region in the plane. The problem here is that no natural boundary conditions are available.

In Section 3 we discuss different possibilities of defining a unique height function. Essentially our approach consists in minimizing some norm of the solution provided that it also satisfies the Poisson equation. In particular we consider the L^2- and H^1-norms. We also show how these two optimization problems may be reformulated as boundary value problems for the biharmonic equation.

As applications, in [14], the eigenvalue problems of the symmetric tensor-block matrix of any even rank and sizes 2×2 is studied. Some definitions and theorems are formulated concerning the tensor-block matrix. Formulas expressing the classical invariants of the tensor-block matrix of any even rank and sizes 2×2 through the first invariants of the powers of this tensor-block matrix are given. As a special case, we consider the tensor-block matrix of the elastic modulus tensors. The canonical representation of the tensor-block matrix is given. Using this representation, we get the canonical forms of the elastic strain energy and the constitutive relations. Besides, a classification of the micropolar linear elastic anisotropic bodies that do not have a center of symmetry is given. In [15], some questions about the parametrization of three-dimensional thin body with one small size under an arbitrary base surface and the changing of transverse coordinate from 1 to 1 are considered. The vector parametric equation of the thin body domain is given. In particular, we have defined the various families of bases and geometric characteristics generated by them.

2. A scattering model

Here we will briefly discuss the mathematical inverse problem to be resolved in order to recover the ground topography height function from radar data. First cylindrical coordinates (r, φ, z) are introduced according to Fig. 1, where it is understood that the aircraft is flying at a constant speed along the z-axis. Further r denotes the distance from a point on the ground surface to the z-axis and φ is the angle between radius vector and a horizontal plane through the z-axis. Then the ground surface may be described by a function $H(r, z)$ through the equation

$$\frac{H(r,z)}{r} - \varphi = 0. \tag{4}$$

When r is large, $-H(r, z)$ is approximately a Cartesian height function. Fig. 2 shows a top view of the same scene. We have also indicated an aspect vector from the aircraft to some point on the ground, forming an angle θ with a vertical plane through the aircraft. Normalized to unit length, the aspect vector is denoted by $\hat{n}$.

Accordingly

$$\hat{n} = \cos\theta\,\hat{r}(\varphi) + \sin\theta\,\hat{z}. \tag{5}$$

Here $\hat{r}(\varphi)$ denotes the cylindrical unit basis vector corresponding to the r-coordinate for the ground point as shown in the Fig. 2. For a point on the ground surface with coordinates (r, φ, z) we obtain, from Eq. (4), the following expression for the ground surface normal $\bar{m}$,

$$\bar{m} = grad\left(\frac{H(r,z)}{r} - \varphi\right) = \frac{\partial(H/r)}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial H}{\partial z}\hat{z} - \frac{1}{r}\hat{\varphi}. \tag{6}$$

Let $\hat{m}$ denote the normalized normal. Then

$$\hat{m} \circ \hat{n} = \left(r\cos\theta\frac{\partial(H/r)}{\partial r} + \sin\theta\frac{\partial H}{\partial z}\right) / \sqrt{1 + \left(\frac{\partial(H/r)}{\partial r}\right)^2 + \left(\frac{\partial H}{\partial z}\right)^2}. \tag{7}$$

Note that (r, φ, z) in Eq. (7) are related to the ground surface point and not to the position of the aircraft.

Let $(z_0, 0)$ be a position of the aircraft and R the distance to some point on the surface. According to Fig. 3 the coordinates (r, z) are then equal to $(z_0 + R\sin\theta, R\cos\theta)$. Next, to obtain a scattering model we will assume that the reflectivity from a ground surface element (see Fig. 4) is

$$\approx \frac{\hat{m} \circ \hat{n}}{R} dR\, d\theta. \tag{8}$$

From Fig. 4, where a vertical plane through $(z_0, 0)$ (the aircraft) and the ground point $(z_0 + R\sin\theta, R\cos\theta)$ is displayed, we conclude that the solid angle $d\Omega$ under which the surface element dS is seen from the antenna is approximately

$$\frac{dR\,\cos\alpha\, Rd\,d\theta}{R^2} = -\frac{\hat{m} \circ \hat{n}}{R} dR\, d\theta.$$

In expression (8) we are consequently assuming that the local reflectivity is proportional to the solid angle occupied by the infinitesimal surface element dS. The total reflected signal $G(R, z_0)$

from all points at a distance R from the antenna may now be obtained by integration over the circle $C(R, z_0) = \{(r, z) : (z - z_0)^2 + r^2 = R^2\}$ in Fig. 3.

$$G(R, z_0)dR = c\int_{-\pi}^{\pi} \frac{\hat{m} \circ \hat{n}(z_0 + R\cos\theta, R\sin\theta)}{R} d\theta dR$$

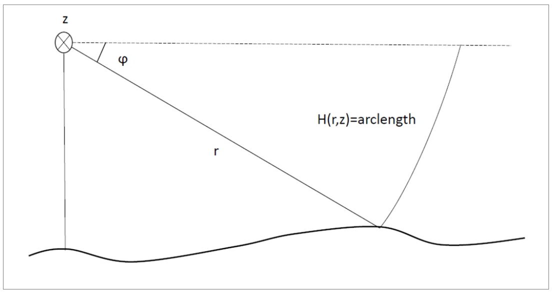

Figure 1. The ground surface measured at a fixed aircraft position.

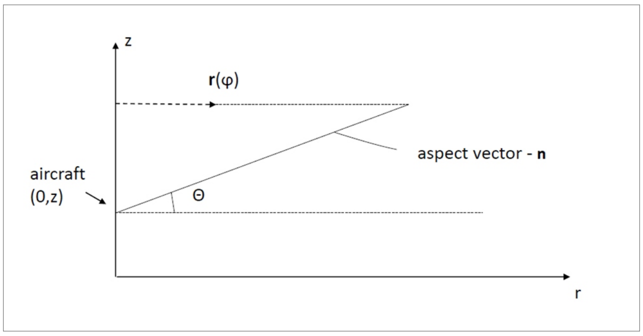

Figure 2. The measuring geometry as seen from above.

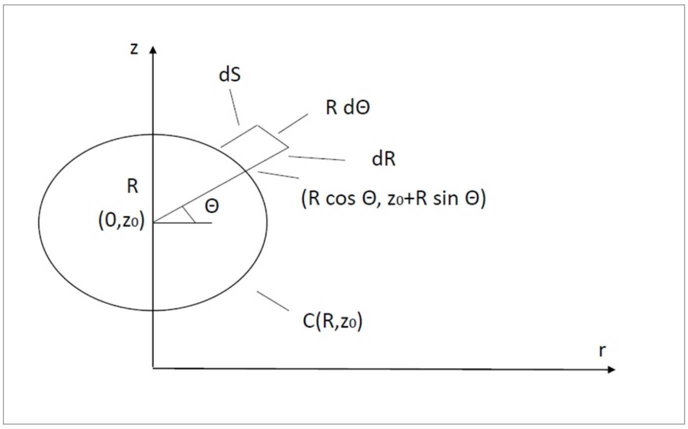

Figure 3. The coordinate system used to describe an infinitesimal surface element, dS.

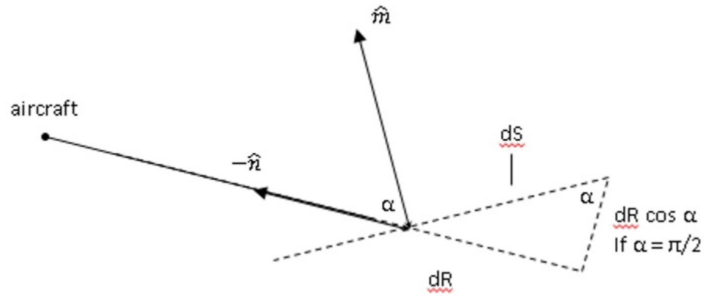

Figure 4. The infinitesimal surface element, dS, as it is seen from the aircraft.

i.e.

$$RG(R, z_0) = c\int_{-\pi}^{\pi} \hat{m} \circ \hat{n}(z_0 + R\cos\theta, R\sin\theta)d\theta. \tag{9}$$

Assuming that $\hat{m} \circ \hat{n}$ is small Eq. (7) may be replaced by

$$\hat{m} \circ \hat{n} = r\cos\theta\frac{\partial(H/r)}{\partial r} + \sin\theta\frac{\partial H}{\partial z}.$$

By inserting this into Eq. (9) we get, after multiplying by R,

$$R^2G(R, z_0) = c\int_{-\pi}^{\pi} \big(rR\cos\theta\frac{\partial(H/r)}{\partial r} + R\sin\theta\frac{\partial H}{\partial z}\big)d\theta.$$

Using the parametrization

$$z = z_0 + Rsin\theta, \quad r = Rcos\theta,$$

this may be rewritten as a curve integral over $C(R, z_0)$, with $dz = R\cos\theta d\theta$ and $dr = -R\sin\theta d\theta$,

$$R^2G(R, z_0) = c\int_{C(R,z_0)} \big(r\frac{\partial(H/r)}{\partial r}dz - \frac{\partial H}{\partial z}dr\big). \tag{10}$$

By applying Green's formula we get

$$R^2G(R, z_0) = c\iint_{D(R,z_0)} \mathfrak{L}(H)(r, z)dzdr, \tag{11}$$

where D is the disc, $D(R, z_0) = \{(r, z) : (z - z_0)^2 + r^2 \leq R^2\}$ and

$$\mathfrak{L}(H) = \frac{\partial}{\partial r}\big(r\frac{\partial}{\partial r}(H/r)\big) + \frac{\partial^2}{\partial z^2}(H). \tag{12}$$

The problem of finding the height function H from radar data $G(r,z)$ may now be divided into two parts.

(a) First solve the integral equation (11) for $\mathfrak{L}(H)(r,z) = f(r,z)$.

(b) Next solve the partial differential equation

$$\mathfrak{L}(H) = f \tag{13}$$

for H. We note that if r is large and if $\hat{m}\circ\hat{n}$ is small it is reasonable to make the approximation

$$\mathfrak{L}(H) \approx \frac{\partial^2 H}{\partial r^2} + \frac{\partial^2 H}{\partial z^2} = \Delta H$$

so that Eq. (13) becomes Poisson's equation. To consider the first problem (a), both members in Eq. (11) are differentiated with respect to R. Then we get

$$\frac{1}{R}\frac{d}{dR}(R^2 G(R,z_0)) = c\int_{-\pi}^{\pi} \mathfrak{L}(H)(z_0 + R\cos\nu, R\sin\nu)d\nu,$$

where the right-hand side is proportional to the average of $\mathfrak{L}(H)$ over the circle $C(R,z_0)$. In [2] an explicit solution is given for this problem of recovering the function $\mathfrak{L}(H)(r,z)$ when the average of $\mathfrak{L}(H)$ is known for all circles $C(R,z_0)$ with center on the z-axis and with arbitrary radius R. The solution formula is

$$\mathfrak{L}(H)^{(F,F)}(\sigma,\omega) \sim |\omega|\big[\frac{1}{R}\frac{d}{dR}\{R^2 G(r,z)\}\big]^{(F,H_0)}(\sigma, \sqrt{\omega^2+\sigma^2}). \tag{14}$$

Here the notation (F,F) means that we have taken the Fourier transform with respect to both the variables and (F,H_0) means that we have taken Fourier transform with respect to the first variable and the Hankel-zero transform with respect to the second. After some calculations Eq. (14) may be rewritten

$$\mathfrak{L}(H)^{(F,F)}(\sigma,\omega) \sim |\omega|\sqrt{\omega^2+\sigma^2}[RG(r,z)]^{(F,H_1)}(\sigma, \sqrt{\omega^2+\sigma^2}). \tag{15}$$

Formula (15) may now be used in order to recover the function $\mathfrak{L}(H)$ in spatial coordinates. Of course, approximating $\mathfrak{L}(H)$ by ΔH we could rewrite Eq. (15) as

$$H^{(F,F)}(\sigma,\omega) \sim |\omega|\frac{1}{\sqrt{\omega^2+\sigma^2}}[RG(r,z)]^{(F,H_1)}(\sigma, \sqrt{\omega^2+\sigma^2}), \tag{16}$$

where H_1 denotes that we have taken the Hankel-one transform with respect to the second variable. Then we could obtain H directly by a two timensional Fourier transform. However, our solution might be expected to have errors caused by, e.g. noisy radar data and errors caused by the particular numerical implementation of the inversion formula (14) (or Eq. (15)) and therefore we would rather prefer to divide the solution procedure into the two steps described above and to use the second step, the solution of Poisson's equation, so that we perform some kind of regularization of the final solution. Note also that by using Eq.(16) as our solution formula we have tacitly assumed periodic boundary conditions for the Poisson equation.

3. Solution concepts for the Poisson equation

In the domain Ω we consider the following boundary value problems for the Poisson equation

$$\Delta u = f(x), \quad x \in \Omega \tag{17}$$

with the boundary conditions

$$u = g \quad \text{on} \quad \partial\Omega, \tag{18}$$

and

$$u = g, \quad \nabla u \cdot \nu = h \quad \text{on} \quad \partial\Omega \tag{19}$$

Finally for Ω a rectangular region in, e.g., the plane

$$\Omega = \{(x,y):\ a < x < b,\ c < y < d\}, \tag{20}$$

there may be the following boundary conditions

$$u(a,y) = u(b,y), \quad u(x,c) = u(x,d), \tag{21}$$

and

$$\begin{cases} u(a,y) = u(b,y), \quad u(x,c) = u(x,d), \\ u_x(a,y) = u_x(b,y), \quad u_y(x,c) = u_y(x,d) \end{cases} \tag{22}$$

Lemma 1 *Let u and w be solutions of Eq.* (17) *satisfying the Dirichlet boundary conditions* (18) *with $g = g_1$ abd $g = g_2$, respectively. Assume $f \in C(\Omega)$, g_1, $g_2 \in C(\partial\Omega)$ and that $\partial\Omega$ is Lipschitz. Then*

$$||u - w||_{\infty,\Omega} \leq ||g_1 - g_2||_{\infty,\partial\Omega}.$$

Hence the Dirichlet problem is well posed in the sence that small pertutbations in the boundary values result in small perturbations in the solution.

We now consider a different way to select a solution to Eq. (17). Here we use a criterion function and optimize this criterion over the set of solutions to the Poisson equation. As discussed in Section 2 the physical interpretation of $u(x,y)$ is a surface function. A possibility is to pick out the smoothest surface (in some sense) that fulfills Eq. (17). We propose to use Sobolev space norms as criterion functions. Denote by $V_{f,i}$ the following set:

$$V_{f,i} = \{u \in H^i(\Omega): \quad \Delta u = f,\ f \in L^2(\Omega)\}, \quad i = 0, 1, 2, \tag{23}$$

where $H^0(\Omega) = L^2(\Omega)$.

The equality $\Delta u = f$ is to be interpreted in the sense of distributions. i.e.,

$$\int_\Omega u\, \Delta\varphi\, dx = \int_\Omega f\varphi\, dx, \quad \forall \varphi \in C_0^\infty(\Omega).$$

Lemma 2 *$V_{f,i}$ is a closed, convex and nonempty set of $H^i(\Omega)$.*

Let α be a multiindex and $\beta_1 > 0$ a given parameter. We consider the following optimization problems:

$$min_{u \in V_{f,0}} ||u||_2^2, \tag{24}$$

and

$$min_{u \in V_{f,1}} ||u||_2^2 + \beta_1 \sum_{|\alpha|=1} ||\partial^\alpha u||_2^2. \tag{25}$$

Theorem 1 *Problems* (24) *and* (25) *have unique solutions u_0 and u_1 respectively.*

We conclude this section by a theorem relating the solution of problems (24) and (25). First we recall the following definition.

Definition 2 *$\Omega \subset \mathbb{R}^n$ is called star-shapet if there exists $x_0 \in \Omega$ such that for all $x \in \Omega$ the set $\{t \in \mathbb{R} : \quad x_0 + t(x - x_0) \in \Omega\}$ is an interval.*

Remark 1 *All convex sets are star-shaped. Rectangles Ω appearing in our applications are thus star-shaped.*

Theorem 2 *Assume that $\Omega \subset \mathbb{R}^n$ is open, bounded and star-shaped. If $u_{1,\beta_1} \in H^1(\Omega)$ denotes the solution of problem* (25) *with the parameter $\beta_1 > 0$ and if $u_0 \in L^2(\Omega)$ denotes the solution of problem* (24)*, then*

$$u_{1,\beta_1} \to u_0 \quad in \quad L^2(\Omega) \quad as \quad \beta_1 \to 0+.$$

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